Lindahl and Equilibrium

by

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Abstract

This paper demonstrates that there is a discrepancy between the ideas expressed in Lindahl (1919) and the current-day definition of Lindahl equilibrium. It describes how the ideas expressed by Lindahl (1919) developed into the equilibrium concept for public good economies that now carries Lindahl's name. The paper also touches on a seemingly forgotten equilibrium concept for public good economies known as ratio equilibrium, and explains that from an axiomatic perspective this equilibrium concept is a better fit with the ideas expressed in Lindahl (1919).

*Department of Economics, University of Oregon, Eugene, OR 97403-1285, USA. E-mail: annev@uoregon.edu. I am grateful to Nick Baigent for suggesting to me that I write this paper, which started as a simple project explaining the discrepancy between the concept known as Lindahl equilibrium and the ideas expressed in Lindahl (1919), but ended up being a detective story. I am indebted to Peter Lambert, Nicholas Sly, and two anonymous referees for their comments which helped me improve the paper, and to Myrna Wooders for advocating the axiomatic technique to study equilibrium concepts in (local) public good economies.

1 Introduction

Lindahl equilibrium and ratio equilibrium are different equilibrium concepts for public good economies. Lindahl equilibrium is based on consumers paying personalized prices for public goods, whereas ratio equilibrium is based on consumers paying personalized shares of the costs of public good production. The two concepts coincide when production of public goods exhibits constant returns to scale, but they differ for more general production processes.

Lindahl equilibrium carries Lindahl's name and it is commonly accepted that it was introduced by Lindahl (1919), and later formalized by Samuelson (1954), Johansen (1963), and Foley (1970). However, in van den Nouweland *et al* (2002) we showed that not Lindahl equilibrium, but ratio equilibrium, which was formalized by Kaneko (1977), accurately represents the cost-share ideas expressed in Lindahl (1919). This discrepancy motivates the current paper. I describe how the literature developed from Lindahl (1919) to Lindahl equilibrium and I discuss the relation between ratio equilibrium and the ideas in Lindahl (1919).

Lindahl (1919) does not contain a mathematical definition of an equilibrium concept, but ideas expressed in text and a picture. These ideas are of individual agents determining their demand for public good based on shares of the cost of public good production. I show that in the literature in which the ideas in Lindahl (1919) were developed into the current-day concept known as Lindahl equilibrium, several things have happened. First, the meaning of the word *price* has evolved from sometimes meaning a total amount to be paid for a certain quantity of a good, into a fixed amount to be paid per unit of a good. Second, at a time when the literature was transitioning from explaining ideas verbally and graphically into using mathematical notation to formalize them, and many economists were unfamiliar with the use of mathematical methods, *price* came to have a very specific meaning

and notation that was introduced for the special case of constant returns to scale came to be used in the definition of Lindahl equilibrium. Third, the assumption of constant returns to scale that was initially carefully mentioned, has been dropped over time. This is acutely relevant given that personalized prices for public good are not equivalent to personalized shares of the cost of its production when the production technology for the public good does not exhibit constant returns to scale.

The paper is organized as follows. In Section 2, I provide the current-day definition of Lindahl equilibrium and some discussion of this definition and its implications. In Section 3, I outline the ideas for an equilibrium concept that were explained in Lindahl (1919). In Section 4, I explain how the literature evolved from the ideas expressed in Lindahl (1919) to the definition of Lindahl equilibrium as a concept based on personalized prices. In Section 5, I describe the axiomatic link between Lindahl (1919) and ratio equilibrium. I conclude in Section 6.

This paper contains many direct quotes. Rather than explicitly attributing each and every quote to its source, whenever there can be no confusion about the source of a particular quote, I will simply surround it by the two signs “ and ”.[1]

2 Lindahl equilibrium and personalized prices

In this section I demonstrate that Lindahl equilibrium is a concept based on personalized prices and discuss some implications of this aspect of the definition. I first provide a definition of a pure public good economy and a general definition of Lindahl equilibrium. I follow that up with descriptions of Lindahl equilibrium taken from three different sources that corroborate that in

[1] In some cases the signs “ and ” also appear in the quotes themselves.

the literature Lindahl equilibrium is indeed defined using personalized prices. I end this section with a discussion of the nature of Lindahl equilibrium and examples that illustrate some potential problems with this concept.

A *pure public good economy* (with one public good and one private good) is a list $E = \langle N; (w_i)_{i \in N}; (u_i)_{i \in N}; c \rangle$ consisting of the following elements. N is a non-empty finite set of consumers. Each consumer $i \in N$ has an initial endowment w_i of the private good, and a utility function $u_i(x, y_i)$ for consumption of amounts x of the public good and y_i of the private good. There is a single producer of the public good and $c(x)$ is the cost in terms of private good for producing an amount x of the public good.

2.1 Definition of Lindahl equilibrium

Lindahl equilibrium is a concept for pure public good economies that mirrors the definition of competitive equilibrium in private-good economies. In a Lindahl equilibrium, each consumer takes prices of all goods as given and demands levels of goods that maximize her utility among the bundles of goods that she can afford given her endowment and those prices. However, unlike in private-goods economies, each consumer is assumed to face a *personalized price* for units of the public good and this price can be different for each consumer. The personalized prices of all consumers are added to find the price at which a producer of a public good can sell its output and the producer determines a profit-maximizing production level given this jointly determined price. In case the producer has a positive profit, this is distributed among consumers according to some exogenously given distribution rule and a consumer's share of the profits is added to her initial endowment when determining her budget constraint. Finally, the market-clearing condition for public goods requires that all consumers demand the same level of public good (as opposed to the condition for private goods that the sum of

demands by all consumers equals available amounts), and that this demand coincides with the producer's profit-maximizing supply.

Different rules for the distribution of profits may lead to different decisions by consumers and thus Lindahl equilibrium is defined with respect to a distribution rule. A distribution rule is a vector $d = (d_i)_{i \in N}$, with $\sum_{i \in N} d_i = 1$, where d_i is the proportion of the profits of public-good production that fall to consumer i and which this consumer can either consume as private good or use to pay for public good. A *d-Lindahl equilibrium* consists of a vector of personalized prices $\mathbf{p}^* = (p_i^*)_{i \in N}$, an amount of public good x^*, and amounts of private good $(y_i^*)_{i \in N}$ such that

1. x^* is a solution to
$$\max_x \left(\sum_{j \in N} p_j^* \right) x - c(x)$$

2 For each $i \in N$, (x^*, y_i^*) is a solution to

$$\begin{aligned} \max_{(x, y_i)} \quad & u_i(x, y_i) \\ \text{subject to} \quad & p_i^* x + y_i \leq w_i + d_i \left(\left(\sum_{j \in N} p_j^* \right) x^* - c(x^*) \right) \end{aligned}$$

This definition of Lindahl equilibrium captures the basic elements that the current-day definitions in the literature have in common. Of course, there are variations in descriptions, notations, and context in different papers, as the following three subsections illustrate.

2.2 Lindahl equilibrium in Mas-Colell *et al* (1995)

Mas-Colell, Whinston, and Green (1995), the standard text used in many graduate programs in economics, describes Lindahl equilibrium on pages 363-4. This description involves, for each consumer i, a market for the public good

"as experienced by consumer i" and a price p_i of this personalized good. The prices p_i may differ across consumers. Given the equilibrium price p_i^{**}, each consumer i decides on the equilibrium amount x_i^{**} of public good (s)he wants to consume so as to solve

$$\max_{x_i \geq 0} \phi_i(x_i) - p_i^{**} x_i,$$

where $\phi_i(x_i)$ is consumer i's utility from consuming an amount x_i of the public good. A firm produces a bundle of I goods - I being the number of consumers - using a fixed-proportions technology such that the level of production of each personalized good is necessarily the same. The firm's equilibrium level of output q^{**} solves

$$\max_{q \geq 0} \left(\sum_{i=1}^{I} p_i^{**} q \right) - c(q),$$

where $c(q)$ is the cost of supplying an amount q of the public good. In equilibrium, the market-clearing condition $x_i^{**} = q^{**}$ has to hold for all i.

Mas-Colell *et al* (1995) assume that the cost function $c(\cdot)$ has a strictly positive second derivative at all $q \geq 0$, and thus they exclude cases where public-good production exhibits constant returns to scale.[2] They state that the type of equilibrium just described "is known as a *Lindahl equilibrium* after Lindahl (1919)" and refer the reader to Milleron (1972) for a further discussion.[3]

2.3 Lindahl equilibrium in Kreps (2013)

Kreps (2013) is a new text whose targeted audience is "first-year graduate students who are taking the standard "theory sequence" and would like to

[2] I am highlighting this feature because we will see in Example 2 that personalized prices can be problematic when marginal costs are not constant.

[3] I will discuss Milleron (1972) in Section 4.

go more deeply into a selection of foundation issues". This text describes Lindahl equilibrium on pages 381-2, as part of a discussion of externalities in the setting of Coase (1960), and states that "Lindahl proposed this equilibrium before Coase (in 1919)". Kreps (2013) takes a full page to describe Lindahl equilibrium, which I have condensed to the following.

Each consumer h maximizes the utility $u^h(\mathbf{x}, \mathbf{z})$ that she accrues, subject to the budget constraint

$$p\mathbf{x}^h + \sum_{h' \neq h} r_{hh'}\mathbf{x}^h \leq pe^h + \sum_f s^{fh}\left[pz^f - \sum_{h'} q_{fh'}\mathbf{z}^f\right] + \sum_f q_{fh}\mathbf{z}^f + \sum_{h \neq h'} r_{hh'}\mathbf{x}^{h'},$$

where p denotes prices for the goods, the $r_{hh'}$ denote transfer prices that record transfers from h to h' made for the choice of $\mathbf{x}^h$ by h, e^h denotes endowment, the s^{fh} denote shares in the profits of the firms, and the q_{fh} denote transfers made from firm f to consumer h made for f's choice of $\mathbf{z}^f$. Also, given prices, a firm f chooses a production plan z^f to maximize the transfer-induced profits $pz^f - \sum_h q_{fh}z^f$. Kreps (2013) states "A Lindahl equilibrium is a vector $(p, q, r, \mathbf{x}, \mathbf{z})$, where: firm f, taking prices as given, maximizes its net-of-transfer profits at $\mathbf{z}^f$; consumer h, taking prices as given, maximizes her preferences at $(\mathbf{x}, \mathbf{z})$, given the budget constraint above; and markets clear, in the usual fashion. N.B., every consumer chooses the full vector $(\mathbf{x}, \mathbf{z})$, and it is a condition of equilibrium that these choices agree."

2.4 Lindahl equilibrium in Oakland (1987)

Oakland (1987) is the chapter on the theory of public goods in *The Handbook of Public Economics*. He refers the reader to Johansen (1963)[4] for the source of the "so-called Lindahl approach", which he describes (verbally) on page 525 as requiring "that individuals be charged (taxed) an amount equal

[4] I will discuss Johansen (1963) in Section 4.

to their marginal valuation times the level of public good supplied". Oakland explains that this approach involves the Marshallian demand curve of each individual for the public good, which "shows the amount of pubic good desired at each constant tax price". These demand curves are summed vertically to find the aggregate marginal valuation for each particular level of the public good and then the resultant curve is intersected with the marginal cost schedule for the public good to find the "efficient level of public good". It is pointed out that "each individual's tax price will vary - the highest price charged to the individual with the greatest marginal valuation".

In a footnote, it is pointed out that, strictly speaking, the approach described "is correct only if the public good is produced under conditions of constant costs", because otherwise "other taxes and subsidies will be required which will affect the underlying demand curves".

2.5 Discussion

The preceding three subsections clearly demonstrate that Lindahl equilibrium is a concept based on prices. Mas-Colell *et al* (1995), Kreps (2013), and Oakland (1987) all, each in their own way, describe consumers who maximize their utilities taking into account a budget constraint in which quantities of public good are multiplied by some per-unit price. They also all include, explicitly or implicitly, a profit-maximization condition, and Kreps (2013) explicitly includes terms related to the the distribution of profits among the consumers, whereas Oakland (1987) hints at the need for a way to distribute profits if production does not exhibit constant returns to scale. In the remainder of this discussion, I will work with the definition in subsection 2.1, which makes all these aspects of Lindahl equilibrium explicit with a minimum of notation.

I discuss some of the issues surrounding Lindahl equilibrium by means of

two simple examples, one of a public good economy with constant returns to scale in public good production, and one with decreasing returns to scale.

Example 1 *Lindahl equilibrium in an economy with constant returns to scale.*

Consider the 2-consumer public good economy with $c(x) = 2x$, $N = \{1, 2\}$, $w_1 = w_2 = 8$, $u_1(x, y_1) = x^{1/4} y_1^{3/4}$, *and* $u_2(x, y_2) = x^{3/4} y_2^{1/4}$. *Obviously, the profit-maximization problem of the producer only has a non-zero solution if* $p_1^* + p_2^* = 2$ *and if this is the case, then all levels of public good result in maximal profits of* 0. *Thus, the Lindahl equilibrium is independent of the way in which profits are distributed among the consumers. Because the utility functions of both consumers are strictly increasing, the budget constraints will hold with equality in equilibrium. Using substitution, we thus can write the utility-maximization problem for consumer 1 as*

$$\max_x \; x^{1/4}(8 - p_1^* x)^{3/4}.$$

From this, it is easily derived that consumer 1 demands an amount $2/p_1^*$ *of the public good. Similarly, we find that consumer 2 demands an amount* $6/p_2^*$ *of the public good. In equilibrium, these amounts have to be equal, which together with* $p_1^* + p_2^* = 2$ *leads to* $p_1^* = 1/2$ *and* $p_2^* = 3/2$. *Thus, we find that* $x^* = 4$, $y_1^* = 6$, *and* $y_2^* = 2$.

This example illustrates that if public good production exhibits constant returns to scale, profit maximization implies that, in equilibrium, the public good is only produced if the sum of the personalized prices is equal to the constant marginal cost. As a result, the maximal profit of the producer is 0 and the equilibrium is independent of the profit distribution among consumers. In such cases there is a clear relationship between a consumer's budget constraint ($p_i^* x + y_i \leq w_i$) and the costs of public-good production,

because $p_i^* x = \frac{p_i^*}{m} c(x)$ for all levels x of public good, where m is the constant marginal cost.

One important instance of constant returns to scale in public good production occurs when the public good is measured in terms of expenditures. However, in cases where the cost function c does not exhibit constant returns to scale, clearly the public good must be measured in some other way than expenditures. Such a case is exhibited in the following example.

Example 2 *Lindahl equilibrium in an economy with decreasing returns to scale.*

Consider the 2-consumer public good economy with $c(x) = x^2$, $N = \{1, 2\}$, $w_1 = 4$, $w_2 = 6$, $u_1(x, y_1) = x + y_1$, *and* $u_2(x, y_2) = 3x + y_2$. *Profit maximization for the producer results in* $x^* = \frac{p_1^* + p_2^*}{2}$ *and the producer has a profit of* $\left(\frac{p_1^* + p_2^*}{2}\right)^2$, *which needs to be distributed among the consumers. For consumer 1's utility-maximization problem to have an interior solution, the consumer's budget line needs to have the same slope as any of her indifference curves;* $p_1^* = 1$. *Similarly,* $p_2^* = 3$ *in a Lindahl equilibrium. This leads to* $x^* = 2$ *and the firm making a profit equal to 4. Thus, with a distribution rule* $d = (d_1, d_2)$ *for profits, we find that* $y_1^* = 4 - 2 + 4d_1 = 2 + 4d_1$ *and* $y_2^* = 4d_2$.

In this example, the profit-maximization condition guarantees a unique level of public-good production in equilibrium. However, the Lindahl equilibrium conditions put no restrictions on the distribution rule and there are many Lindahl equilibria with different outcomes for the consumers. This multiplicity (and the need to re-distribute profits) stems from the fact that the consumers do not take the actual cost $c(x)$ of production into account in their utility-maximization problem, but consider a linear budget constraint $p_i^* x + y_i \leq w_i + d_i \left(\left(\sum_{j \in N} p_j^* \right) x^* - c(x^*) \right)$ that is obtained from a fictional personalized per-unit price p_i^* for the public good.

There is a large literature related to Lindahl equilibrium, mainly focussed on the efficiency and re-distribution properties that our last example illustrates.[5] It is not my goal to cover this literature. Instead, I am interested in the personalized prices-based nature of Lindahl equilibrium and the origins of this particular feature of the concept.

3 Lindahl (1919)

In this section I discuss the ideas for an equilibrium expressed in Lindahl (1919)[6] and show that these ideas involve consumers anticipating having to pay shares of the cost of public good production. Of course, in general, this results in budget sets that are not linear and therefore are very different from the ones obtained when consumers are assumed to face fixed per-unit prices for public good. The ideas in Lindahl (1919) are more closely related to the costs of public good production than the current-day definition of Lindahl equilibrium expresses.

As was customary at the time, Lindahl (1919) contains very little mathematical elaboration. Particularly, it does not contain an explicit mathematical definition of an equilibrium concept, but expresses ideas verbally and graphically. The discussion below reflects these aspects of the paper.

Lindahl states "We may begin by assuming that there are only two categories of taxpayers [...]. Within each category all individuals must pay the same price for their participation in public good consumption. The problem is the relative amount of the two prices, *i.e.* the distribution of the total cost of the collective goods between the two groups." and "One party's demand

[5] See, for example, Silvestre (2003).

[6] To be precise, to the translation thereof in *Classics in the Theory of Public Finance*, Eds. R. Musgrave and A. Peacock, 1958

for certain collective goods at a certain price appears from the other party's point of view as a supply of these goods at a price corresponding to the remaining part of total cost: for collective activity can only be undertaken if the sum of the prices paid is just sufficient to cover the cost." We see in these two quotes that price is used in the meaning of 'part of total cost', which implies that Lindahl uses the term *price* not as we use it today in economics - meaning as a variable to be multiplied by a quantity in order to figure out how much to pay - but as a total amount to be paid.

This interpretation is confirmed when Lindahl continues discussing in terms of shares of total cost. He states that "the question of distribution really means how big a share of certain total costs each party has to bear" and his idea is that "since the extent of collective activity is not given *a priori*, but is one of the variables of the problem, the absolute amount of taxation has to be determined at the same time as its distribution" and thus "the extent of collective activity desired by the tax payers becomes largely decisive for their cost share". Thus, in Lindahl's view the issue is the determination of the shares of the cost of public good provision to be paid by each of the parties. Moreover, because the total costs depend on the amount of public good provided, the determination of the cost shares and the level of provision must be determined simultaneously.

Lindahl proceeds to address this simultaneous determination and illustrates "the manner in which equilibrium is established" with a diagram, which is re-produced in Figure 1.[7] As Lindahl explains, in this figure the variable on the horizontal axis represents "the relative share of one party (A) in total cost at various distribution ratios. At point O party A pays nothing at all towards total cost, leaving the entire burden to the other party, B. The further we move away from O, the greater becomes A's share and the

[7]In this re-production I have left out a few markers that Lindahl uses in a part of his subsequent discussion that I am not covering.

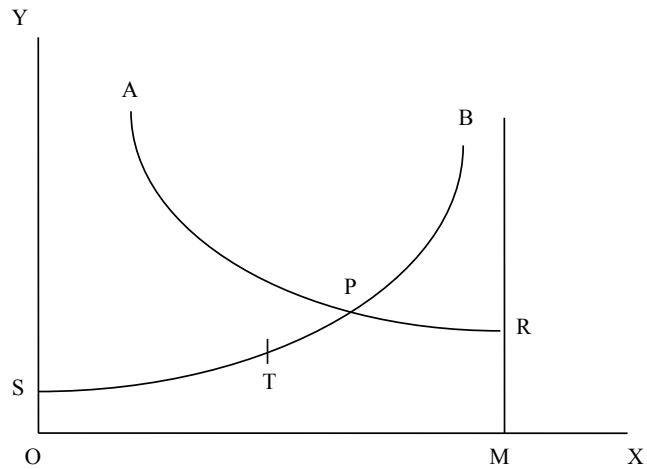

Figure 1: Lindahl's Diagram

smaller B's. At point M the situation is completely reversed; A carries the whole burden and B none." The variable on the vertical axes is "the amount of public expenditure which each party is prepared to sanction at the various distribution ratios." The two curves labeled A and B in the figure represent the "monetary expression of the marginal utility of total public activity for the two parties" and because "demand rises up to the point where marginal utility equals price," these curves represent "demand for public goods" as a function of the "part of public expenditure" that each party has to shoulder.

Lindahl states his equilibrium idea as follows: "The intersection point of the two curves indicates the only distribution of costs at which both parties agree on the extent of public activity." As part of his explanation, he offers "Let us suppose, for example, that the two parities initially agree to split the costs in equal parts. A provisional equilibrium will be established at point T. But only half of A's demand is satisfied and this party will insist on an expansion of public activity. Party B can agree to this only if it can secure a more favourable distribution of costs, and A will have to face the

fact that it must take on a greater share of the cost burden. [...] the shift of the equilibrium position towards P continues smoothly only so long as A's growing sacrifice - and it grows in a double sense, by virtue of both the increase in public expenditure and of the increase of A's share in the cost - is more than compensated by the greater utility due to the expansion of collective activity."

Lindahl's description of equilibrium thus involves the parties weighting their demand for public good against the share of public expenditure that they have to shoulder. When considering their demand for public good, the parties take into account their share of the cost of public good provision and how this share may have to change if they demand alternate levels of public good.

Lindahl (1919), apparently far ahead of his time,[8] provides an algebraic illustration of the preceding discussion. This illustration starts as follows: "Party A contributes fraction x to the total public expenditure and party B hence $1-x$; y is the amount of public expenditure expressed in money; $f(y)$ and $\phi(y)$ are the monetary expressions of the total utility of this expenditure for A and B respectively. Curve A then has the equation

$$f'(y) = x$$

where $f'(y)$ is the utility increment accruing to A from the last unit of money spent and x is the proportion in which A has contributed to this money unit." This shows that Lindahl envisions the parties paying shares of the costs of public good production, and that in this particular illustration the public good is measured in terms of its expenditures.

[8]This is evidenced by the discussion of Novick (1954) and Enke (1955) in Section 4. Lindahl expresses that he is indebted to Knut Wicksell for the algebraic illustration.

4 From Lindahl (1919) to Lindahl equilibrium

As we saw in Section 2, in a Lindahl equilibrium the parties base their demand for public good on personalized prices which are not necessarily a realistic measure of the cost at all levels of public good production. In the current section I explain how the literature has evolved from the ideas expressed in Lindahl (1919) to the definition of Lindahl equilibrium.[9]

To help the reader to follow the flow of ideas throughout the literature that I am describing below, I include an appendix with a reference graph that shows the relations between the papers I address. This reference graph encompasses all the papers that are relevant to the development of Lindahl equilibrium into a concept based on personalized prices. As I will explain below, this feature of Lindahl equilibrium has become established by the time of publication of Milleron (1972), which is why that paper is shown as the most recent one in the reference graph, whereas the term "Lindahl equilibrium" was coined in Foley (1970). At the base of the reference graph (i.e., papers that are not referenced by other papers in the graph) are Lindahl (1919), which is generally believed to be the source of Lindahl equilibrium, and also Bowen (1943), Samuelson (1954), and Novick (1954). Bowen (1943) is included because Samuelson (1955) draws on it for certain aspects of what is to become Lindahl equilibrium. Samuelson (1954), interestingly, does not refer to Lindahl (1919); the reason for this becomes clear in Samuelson (1955), where we read that he did not have access to Lindahl (1919). Novick (1954) is not relevant for the development of Lindahl equilibrium per se, but is included because, as I will explain below, this paper and Enke (1955) provide

[9] One matter that I have to deal with is that in the earlier literature references to others' work are not always explicit and when they are explicit, the references are generally listed in full in the text or in a footnote. In what follows below, I acknowledge references as precisely as I can and when a full reference is available, I acknowledge it in a modern-day format. This has changed the way these references appear in the quotes from older papers.

background information about the use of mathematics in economics that turns out to be relevant for the purpose of the current paper.

In the remainder of this section, I discuss the papers included in the reference graph in chronological order (from older to newer) and show how these papers build on one another. The discussion will reflect the gradual progression of the literature from verbal and graphical explanations of ideas to their mathematical formalizations.

4.1 Musgrave (1939)

Musgrave (1939) pretty closely follows the ideas in Lindahl (1919), and writes about "relative distribution of tax shares", "portion of the total cost", and "percentage of the total cost". He states "a final agreement between the two is reached at a volume of the public services at which the sum of the percentage shares which both are willing to contribute equals 100 per cent of the cost of supplying these services" (pages 215 and 216). On page 217, Musgrave starts to use the terminology *price* and *per cent* interchangeably, for example when he states "B's demand price (requesting A to contribute SH per cent) falls below A's supply price (his willingness to contribute SD per cent)". Musgrave posits "The preceding exposition of the theory agrees with Lindahl's version, which both in refinement and conciseness of argument is superior to those of other authors of the school" and proceeds to discuss the "pricing process" in terms of "percentages" and "ratio of cost distribution".

Musgrave (1939) verbally describes an equilibrium in which two parties are motivated by shares of the costs of public good provision, and he uses the word prices interchangeably with words like percentages, ratios, and shares. Clearly, he does not use the word price in the sense in which we would use it today - namely as a monetary amount per unit of the public good that is invariable with the level of provision of said good. This difference will

become critical when other authors start formalizing the ideas because then prices will notationally have a very specific meaning.

4.2 Bowen (1943)

Bowen (1943) does not refer to earlier work by Lindahl or Musgrave, but I include it here because, as we will see later, Samuelson (1955) draws on this paper and points out that it shows a large similarity with Lindahl (1919). Bowen (1943) recognizes that it is important "to establish meaningful units in which quantities of social goods may be measured". He states that one approach is to measure these "simply in terms of their money cost", but that in many cases "physical units would be preferred" and in such cases "increasing, constant, or decreasing cost may apply, whereas if cost units are used, only constant cost may apply".

Continuing, Bowen states that "assuming a "correct" distribution of income, each person's taste [for public good] can be expressed by a curve indicating the amount of money he would be willing to give up in order to have successive additional quantities made available in the community". He includes a "Figure 1 for a community which is assumed to contain three persons", a copy of which appears as Figure 2 in the current paper. The figure shows the "curves of individual marginal substitution" (MS) for each of the three consumers between a social good (X) and "other goods (money)". For each quantity of the public good, the marginal rates of substitution of the three individuals are added to give the "curve of total marginal substitution" (TMS). Bowen likens this curve to the "familiar curve of total demand" and then states "One of the cardinal principles in determining the output of an individual good is that price should equal cost, i.e. average cost or marginal cost, whichever is lower. This implies that the ideal output is indicated by the point of intersection between the curve of total marginal substitution and

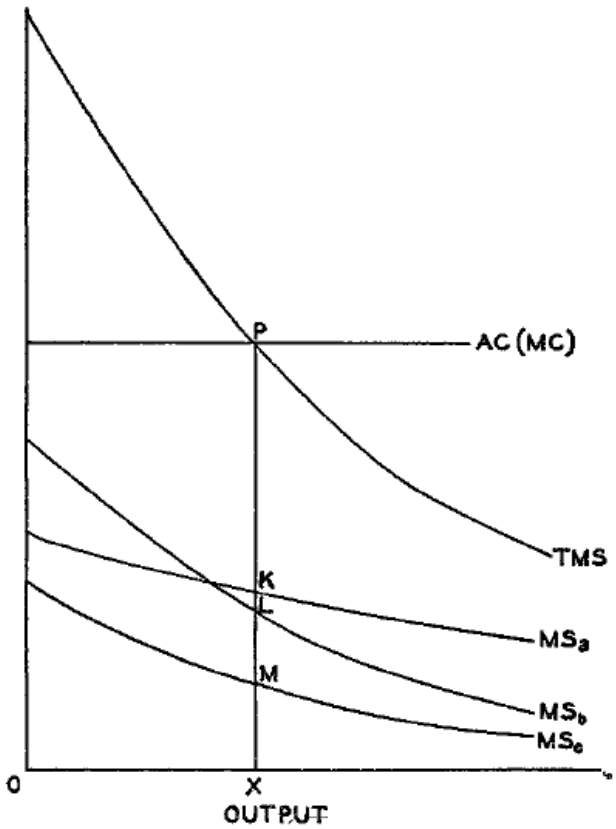

Figure 2: Figure 1 in Bowen (1943)

the appropriate cost curve." Bowen then simply continues to refer to this intersection point as "the optimum output of social goods" without explaining or motivating why principles for individual goods should be extended like this to social goods.

Bowen acknowledges that the cost curve may show increasing or decreasing cost, but that his Figure 1 assumes constant cost. Thus, he explicitly allows for the marginal cost of public good to not be constant, but a reader who is mainly looking at his figure may miss this point.

Bowen (1943) shows similarity to Lindahl (1919) with the statement that "each individual's preference will depend upon [...] the cost *to him* of different amounts of" the social good and that this "will depend partly on the total cost to the community of different amounts and partly on the contemplated distribution of that cost among different individuals" (page 34, italics in original). Bowen concludes from this that each individual will want a quantity of the social good "at which *his* marginal rate of substitution is equal to *his* marginal cost".

The similarity to Lindahl (1919) ends here, however, because Bowen stipulates, for general cost cases, that "the cost must be raised by means of a tax levied upon each individual in the form of a "price" per unit of the social good", where "the price is to remain constant regardless of output" and is "equal to his marginal rate of substitution at the particular amount of the good being produced".

4.3 Samuelson (1954)

Samuelson (1954) does not have any explicit references, but starts with a statement "Except for Sax, Wicksell, Lindahl, and Bowen, economists have rather neglected the theory of optimal public expenditure", and also mentions "published and unpublished writings of Richard Musgrave", "theories

of public finance of the Sax-Wicksell-Lindahl-Musgrave type", and "Bowen's writings of a decade ago".

Samuelson (1954) contains what appears to be the first attempt to formalize a theory of public expenditure using mathematical notation. He states that "in simple regular cases" a "best state of the world" is defined mathematically by the "marginal conditions"

$$\frac{u^i_j}{u^i_r} = \frac{F_j}{F_r} \qquad (i =, \ldots, s;\ r, j = 1, \ldots, n) \tag{1}$$

$$\sum_{i=1}^{s} \frac{u^i_{n+j}}{u^i_r} = \frac{F_{n+j}}{F_r} \qquad (j = 1, \ldots, m;\ r = 1, \ldots, n) \tag{2}$$

$$\frac{U_i u^i_k}{U_q u^q_k} = 1 \qquad (i, q =, \ldots, s;\ k = 1, \ldots, n) \tag{3}$$

where $\{1, 2, \ldots, i, \ldots, s\}$ is the set of individuals, each with a utility function $u^i(X^i_1, \ldots, X^i_{n+m})$ for consumption of "private consumption goods" $X_1, \ldots, X_n$ (with $X_j = \sum_1^s X^i_j$) and "collective consumption goods" $X_{n+1}, \ldots, X_{n+m}$ (with $X_{n+j} = X^i_{n+j}$ for every individual i), with partial derivatives $u^i_j = \frac{\partial u^i}{\partial X^i_j}$, where F models a "convex and smooth production-possibility schedule relating totals of all outputs, private and collective; or $F(X_1, \ldots, X_{n+m}) = 0$, with $F_j > 0$ and ratios F_j/F_n determinate and subject to the generalized laws of diminishing returns to scale", and where $U = U(u^1, \ldots, u^s)$ is a social welfare function with positive partial derivatives U_j. Samuelson explains that the set of conditions (2) are the new element added and that these conditions constitute "a pure theory of government expenditure on collective consumption goods". Note that Samuelson's description allows for production of public goods to not exhibit constant returns to scale and that his marginal conditions do not depend on prices.

Samuelson (1954) continues by stating that "the involved optimizing equations" can be solved using "competitive market pricing" under some conditions, which include "the production functions satisfy the neoclassical

assumptions of constant returns to scale" and "all goods are private". Only for cases satisfying his conditions does Samuelson (1954) introduce prices into his analysis: "We can then insert between the right- and left-hand sides of (1) the equality with uniform market prices p_j/p_r and adjoin the budget equations for each individual $p_1X_1^i + p_2X_2^i + \cdots + p_nX_n^i = L^i$ where L^i is a lump-sum tax for each individual so selected as to lead to the "best" state of the world". Thus, we see that Samuelson (1954) uses prices in his analysis, but that their use is limited to cases of private goods and constant returns to scale. Samuelson explicitly states that the use of prices cannot be extended to cases where collective consumption is not zero: "*However no decentralized pricing system can serve to determine optimally these levels of collective consumption.*" (emphasis in original).

4.4 Novick (1954) and Enke (1955)

There is an interesting exchange related to Samuelson (1954) that may help explain how personalized per-unit prices made their way into the definition of Lindahl equilibrium in subsequent literature. The exchange concerns the rising use of mathematics in economics, which has as a side effect that "many able economists, especially the older and more experienced ones, cannot comment on some published ideas because they cannot "read" them" (quoted from Enke (1955) page 131). The exchange starts with Novick (1954), who argues that "the mathematically uninitiated jump from theory to proof to application without recognizing the intervening steps that usually must be worked out" and who calls for "a broader discussion of these limitations of the mathematical expressions currently used increasingly in the social sciences." Enke (1955) states that Novick's paper "has rather unexpectedly been made the center of controversy" and writes his paper as "a rejoinder" to this controversy, which is left unspecified and apparently takes place in

the general public domain. He goes on to consider Samuelson (1954) as an illustration because it "follows hard upon some of the wise precepts suggested in reply to Novick" and "Samuelson himself cites his work as an example of the uses of mathematical economics." Enke (1955) states about Samuelson (1954) "it is unnecessarily unintelligible to most people. Many economists, interested in public finance and welfare, will want to understand what anyone of Samuelson's reputation has to contribute. Frustration will be their lot."

Hence, at the time of Samuelson's (1954) attempt to formalize a theory of public expenditures, economists in general were not very comfortable with the use of mathematics. It is therefore not hard to imagine that, somewhere along the way, someone may have picked up on Samuelson's easier-looking expressions that include prices and overlooked the fact that Samuelson intended these to be valid for only very specific cases. A study of subsequent literature demonstrates exactly what happened.

4.5 Samuelson (1955)

Samuelson (1955) is apparently written partly in response to the criticism raised in Enke (1955) and "presents in terms of two-dimensional diagrams an essentially equivalent formulation" of the ideas in Samuelson (1954). Samuelson (1955) is the first instance I find of the use of the term "public consumption good", which is defined as a good for which "each man's consumption of it [...] is related to the total [...] by a condition of *equality* rather than of summation." Samuelson then explicitly sets the consumption of public good by individuals 1 and 2 equal ($X_2^1 = X_2^2 = X_2$) and relates this to the total consumption $X_1^1 + X_1^2 = X_1$ of private good by means of a "production-possibility or opportunity-cost curve" that "relates the total productions of public and private goods in the usual familiar manner: the curve is convex from above to reflect the usual assumption of increasing relative marginal

costs (or generalized diminishing returns)". A copy of Samuelson (1955)'s Chart 3 (page 351) appears as Figure 3 in the current paper. It is clear from this figure that Samuelson (1955) does not assume constant returns to scale in public good production.

Samuelson (1955) proceeds to derive graphically the tangency conditions that are necessary for Pareto optima[10] and the set of utility possibilities in Pareto optimal points (the curve labeled pp in Chart 4, a copy of which appears as Figure 4 in the current paper). This together with some contours of a "social welfare function" illustrates the "best configuration for this society". Samuelson verbally explains that "this final tangency condition" has the interpretation that "The social welfare significance of a unit of any private good allocated to private individuals must at the margin be the same for each and every person" and "The Pareto-optimal condition, which makes relative marginal social cost equal to the sum of all persons' marginal rates of substitution, is already assured by virtue of the fact that bliss lies on the utility frontier". Note that Samuelson does not advocate for a specific Pareto optimal point, but shows how to find one that is best for society if a particular social welfare function is given. His work is related to identifying a level of public good that maximizes social welfare, and not to how a society can arrive at that level through decisions by individuals.

After completing "the graphical interpretation of my mathematical model", Samuelson (1955) relates his graphical treatment to earlier work by Lindahl and Bowen, and he does so by means of Chart 5 (page 354), which I include in the current paper as Figure 5. Samuelson (1955) derives an "MC curve" with "MC measured in terms of the numeraire good", by plotting "the absolute slope" of the opportunity-cost curve "against varying amounts of the public good". Doing similarly for the individual indifference curves to obtain

[10] Samuelson uses the point E and the curve $C'D'$, which represents an indifference curve for one of the consumers, for this derivation.

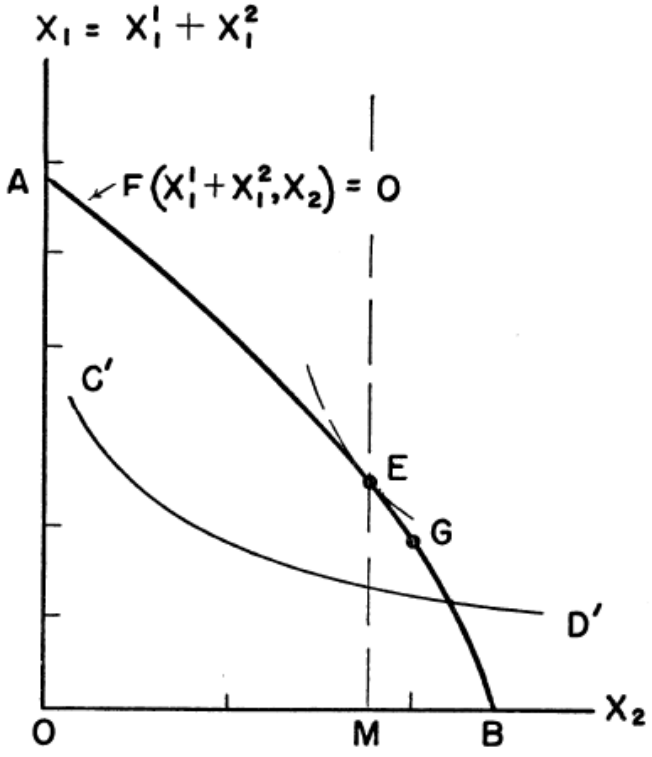

Figure 3: Chart 3 in Samuelson (1955)

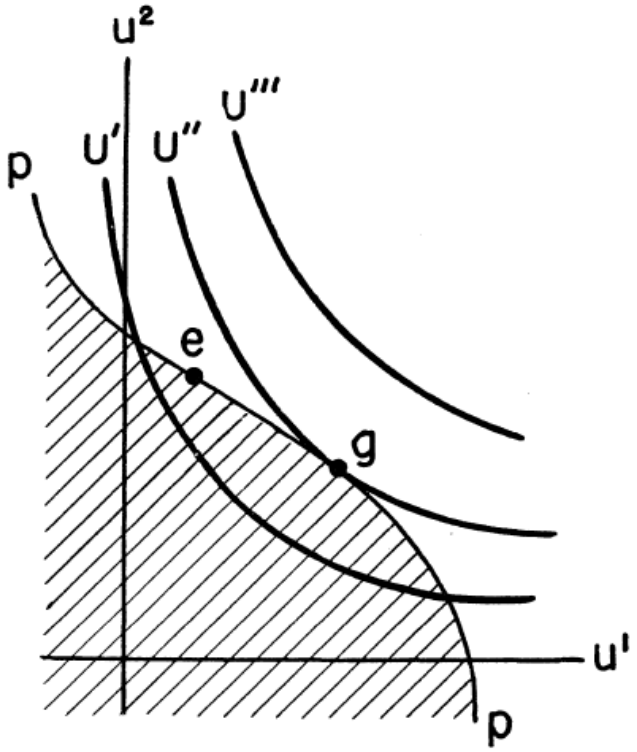

Figure 4: Chart 4 in Samuelson (1955)

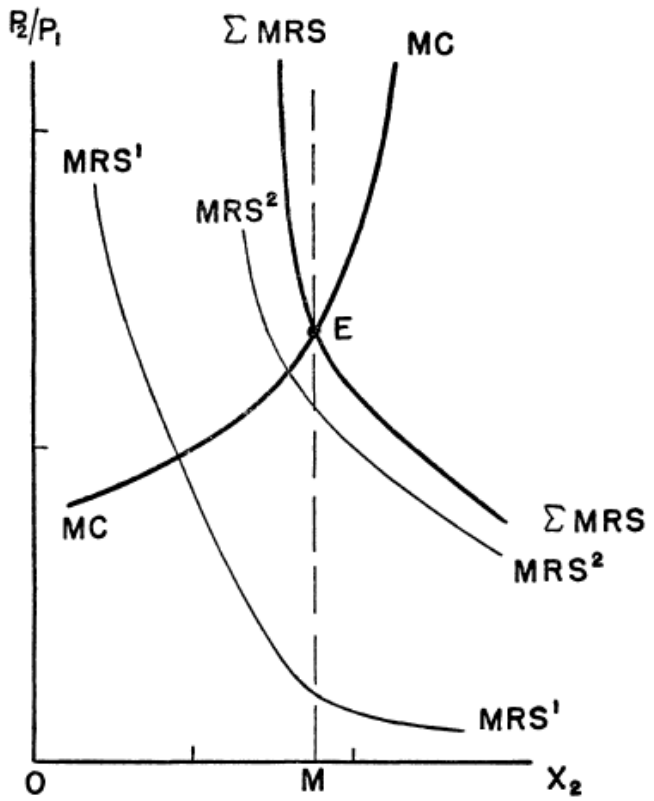

Figure 5: Chart 5 in Samuelson (1955)

individuals' MRS curves, Samuelson then arrives at a picture that shows the addition of the MRS curves (added in the dimension of the private good) and he intersects the result with the MC curve to obtain "equilibrium". Note that Samuelson is using the word equilibrium in a context where he is characterizing a level of public good that satisfies the tangency conditions necessary for Pareto optima. His Chart 5 does not address the individuals' numeraire good consumption levels.

Samuelson acknowledges that "except for minor details of notation and assumption" his Chart 5 (Figure 5 in the current paper) is identical with Figure 1 in Bowen (1943) (Figure 2 in the current paper) and goes on to say "anyone familiar with Musgrave (1939) will be struck with the similarity between this Bowen type of diagram and the Lindahl 100-per-cent diagram reproduced by Musgrave (1939)."[11] However, as we have seen, Figure 1 in Bowen (1943) is for the special case of constant returns to scale. Moreover, the similarity between the "Bowen type of diagram" and "the Lindahl 100-per-cent diagram" is only valid in this special case, as becomes clear in the next subsection.

4.6 Musgrave (1959)

Musgrave (1959) states that his discussion of Lindahl (1919) goes back to Musgrave (1939)[12] and discusses two taxpayers who agree to "contribute certain percentages of the total cost" of "whatever volume of social goods is supplied". Musgrave illustrates his discussion with two figures captioned "Bowen model" and "Lindahl model", respectively (page 75), a re-production

[11] Referring to Lindahl (1919), Samuelson states (Footnote 8) that he has "not had access to this important work".

[12] Actually, he has the year wrong in his own reference - in footnote 1 on page 74 he says 1938 - but this is clearly a typo.

of which is included as Figure 6 in the current paper.[13] Both figures have "units of social goods" on the horizontal axis, and the first figure has "combined unit price" on the vertical axis, while the second has "per cent of cost contributed" on the vertical axis. Musgrave describes how the second picture can be obtained from the first by looking at percentages of S paid by a rather than absolute amounts, which leaves the curve for a in place and basically mirrors the curve for b in the horizontal 50% line. He states that "the resulting price determination" is shown in both figures, where the amount E is obtained in the graph on the left as the quantity where the sum of curves aa and bb intersects SS, and in the graph on the right as the quantity at which the curves $a1a1$ and $b1b1$ intersect. He states, apparently as an afterthought and very casually, "SS is the supply schedule of social goods that we assume are produced under conditions of constant cost" (page 76). In a footnote he then remarks that his figures "may be adapted to conditions of increasing cost", but he does not elaborate.

It is unclear how Musgrave intends to adapt his figures to cases of increasing cost. It does not take much effort to see that if we were to draw a graph like the one on the right in Figure 6 but with an increasing cost function, and we were then to construct a graph like the one on the right that corresponds to this increasing-cost example, then to fit the scales of "per cent cost contributed" by A and B, we would have to interpret cost as total cost and then the two graphs may not predict the same quantity of social goods anymore. Thus, Musgrave effectively links prices and percentages of costs only in cases of constant returns to scale. However, in the discussion of Lindahl's work that follows the two figures, Musgrave continues to use the terms *price*, *per cent of the cost*, and *cost share* interchangeably, without limiting himself to cases of constant cost.

[13] I have left several lines out that are unnecessary for my purposes.

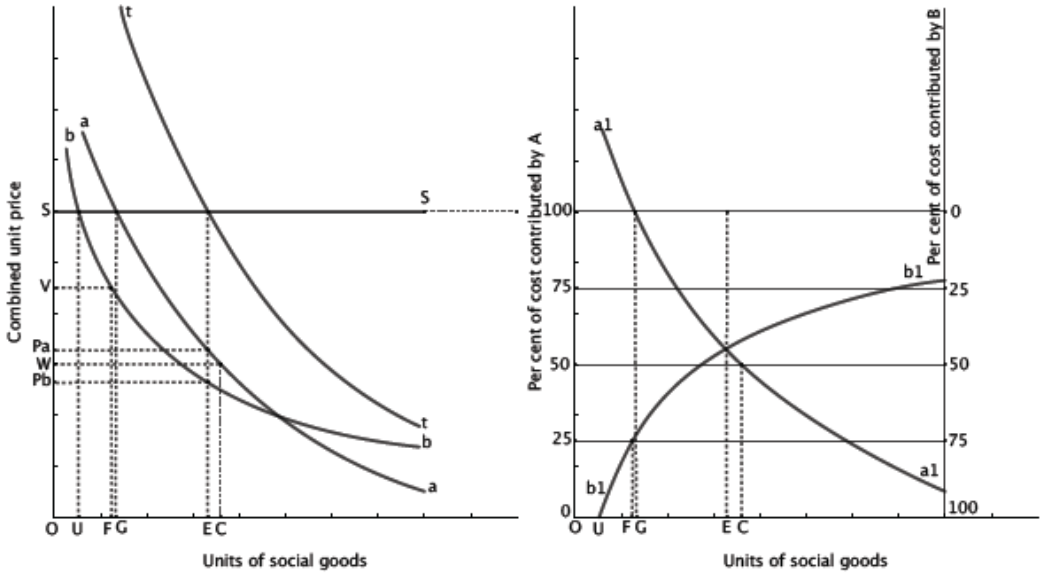

Figure 6: Figures 4-1 and 4-2 in Musgrave (1959)

4.7 Johansen (1963)

Johansen (1963) "is an attempt to present Lindahl's solution in terms of more modern welfare-theoretical concepts and thereby to bring out some new aspects of the solution" and in passing suggests that "the Lindahl solution has not been quite satisfactorily presented" in Musgrave (1959).

Johansen (1963) considers two parties with private consumption and "the amount of public expenditures G", all "measured in monetary units" and he assumes "prices to be fixed". He uses "utility functions [...] with private and public consumption entering side by side" as were used in Samuelson (1954). Johansen (1963) follows Lindahl and has the two parties pay fractions h and $1-h$, so that the "absolute burdens levied" are hG and $(1-h)G$. It is important to note that Johansen measures public good in its expenditures and thereby limits himself to a setting where cost functions for public good exhibit constant returns to scale. Thus, he obtains budget constraints that are linear in G.

Johansen derives the Lindahl diagram using these budget constraints and indifference curves of the utility functions and states (page 349): "The solution is [...] analogous to the determination of an equilibrium in a perfectly competitive market: The equilibrium price is the price at which buyers and sellers "agree" upon the quantity to be traded, when both sides of the market consider the price as given." In his mathematical exposition of "the Lindahl solution", Johansen (1963) derives that when person A maximizes his utility F_A with respect to public expenditures G, while considering A's "income" R_A and "price" h as given, the optimality condition $\frac{\partial F_A}{\partial G} = h\frac{\partial F_A}{\partial X}$ determines the amount of public expenditure wanted by person A (G_A) as a function of R_A and h. He states that this "may be compared with an ordinary demand function with h being the price" and explains that the "fraction of the total public expenditure" paid by a person depends on "the "equilibrium value" of h".

From the preceding quotes, we see that Johansen (1963) describes the fraction h of public expenditures using the word price in the meaning of constant per-unit price. The two words can be used interchangeably because he measures public good in terms of its expenditures. However, it is not hard to imagine how this use of the word price may get extended to situations where public good is measured in some sort of physical quantities and the marginal costs may no longer be constant.

4.8 Foley (1967)

Foley (1967) acknowledges the earlier work by Lindahl, Musgrave, Samuelson, and Johansen, but he defers "all references to sources" to an appendix and thus it is not possible to tell to whom he attributes which parts of his analysis. This paper introduces and studies "a straightforward generalization of ordinary competitive equilibrium" that Foley names "public competitive equilibrium". A public competitive equilibrium consists of three elements: (a) one bundle of public goods and for each agent a bundle of private goods, (b) per-unit prices for all goods - private and public - and (c) a vector of lump-sum taxes for the agents that add up to the cost of the public-goods bundle. Foley (1967) assumes that "production follows a rule of convexity" and writes that "it does not matter very much what units are used to measure the production of public goods as long as the measurement allows each person to decide which of two bundles (including both public and private goods) he prefers". Thus Foley (1967) deviates in an important way from previous literature because he has per-unit prices for public goods without necessarily having constant returns to scale in production.

4.9 Samuelson (1969)

Samuelson (1969) is "a brief review of the analysis" of the "unifying theory of optimal resource allocation to public or social goods and optimal distribution of tax burdens" previously presented in Samuelson (1954 and 1955), an analysis of which "the partial equilibrium analysis of Lindahl and Bowen" is "a special case". Samuelson (1969) states that with the "notable exception" of Johansen (1963), "a number of the theory's essential points have [...] been misunderstood by writers in the field". The paper proceeds to describe "equilibrium" in terms of "*pseudo*-tax-prices" for the public goods, which are personalized per-unit prices that are independent of the actual levels of public goods produced. Unfortunately, any restrictions on the costs for production of public goods do not appear in the body of the paper, but are relegated to an appendix.

4.10 Foley (1970)

Foley (1970) is the paper in which I find the first use of the name "Lindahl equilibrium". This is a very different paper from Foley (1967) in two important respects: In an apparent change of heart, Foley now defines a different equilibrium and also adds a condition of constant returns to scale. Foley (1970) acknowledges "the rehabilitation and reconstruction of Lindahl's (1919) quasi-demand solution to the taxation problem", which he attributes to Johansen (1963) and Samuelson (1969)[14], and provides the following definition:

A *Lindahl equilibrium* with respect to $w = (w^1, \ldots, w^n)$ is a feasible allocation $(x; y^1, \ldots, y^n)$ and a price system $(p_x^1, \ldots, p_x^n; p_y) \geq 0$ such that

[14]Foley (1970) references an "unpublished manuscript" by Samuelson without a date. However, this manuscript has the same title as Samuelson (1969) as referenced in Malinvaud (1971).

(a) $\left(\sum_{i=1}^{n} p_x^i; p_y\right)\left[x; \sum_{i=1}^{n}(y^i - w^i)\right] \geq \left(\sum_{i=1}^{n} p_x^i; p_y\right)(\bar{x}; \bar{z})$ for all $(\bar{x}; \bar{z}) \in Y$

(b) if $(\bar{x}^i; \bar{y}^i) \succ_i (x; y^i)$ then $p_x^i \cdot \bar{x}^i + p_y \cdot \bar{y}^i > p_x^i \cdot x + p_y \cdot y^i = p_y \cdot w^i$.

Here, $x = (x_1, \ldots, x_m)$ denotes a vector of public goods, $y^i = (y_1^i, \ldots, y_k^i)$ denotes a vector of private goods consumed by a consumer i, z denotes a vector of private goods used in production, w^i denotes initial endowments of private goods of consumer i, and Y denotes "the set of all technically possible production plans" for producing public goods from private goods. The first condition on the production set Y (B.1. on page 67) is "Y is a closed, convex cone". Thus, Foley's (1970) definition of Lindahl equilibrium is limited to public goods whose production technologies exhibit constant returns to scale.

4.11 Milleron (1972)

Milleron (1972) is a survey article with some "original contribution", and it takes elements from many of the papers that I have covered up to now. Discussing Samuelson (1954 and 1955), Milleron asserts that the main result of those papers was to show that "with any Pareto optimal situation may be associated a system of "prices", [...] the price paid for the public goods being "*personalized*." These personalized prices [...] may be interpreted as a contribution of each agent to the production of each public good. The sum of contributions is then equal, for each public good, to the production price of this good." Milleron (1972) proceeds by deriving "Samuelson prices" under "a rather general set of assumptions". These assumptions, however, do not include constant returns to scale in production, which was a condition Samuelson was careful to include when invoking the existence of prices for public goods.

Milleron (1972) refers to Johansen (1965) for a discussion of the concept of Lindahl equilibrium, and states "Lindahl's idea was that, for a given

production price of a public good, it is meaningful to define personalized prices such that the sum of these personalized prices is equal to the production price. Thus, it is possible to define the "demand" for public good by each consumer as a function of the corresponding personalized price". On page 439, Milleron (1972) provides a definition of Lindahl equilibrium that includes personalized per-unit prices.

I conclude that at this point in the literature, Lindahl equilibrium has become established as a concept based on personalized per-unit prices for public goods.

5 Lindahl (1919) and ratio equilibrium

In this section, I describe the relation between Lindahl (1919) and ratio equilibrium. I start by providing a definition of ratio equilibrium, then discuss the axiomatic link between the ideas expressed in Lindahl (1919) and ratio equilibrium, and conclude this section with a discussion of relations between ratio equilibrium and Lindahl equilibrium.

5.1 Ratio equilibrium

A ratio equilibrium, first defined in Kaneko (1977), consists of a set of personalized ratios - one for each consumer - and for each consumer a consumption bundle of public and private good amounts. For each consumer i, her ratio r_i determines a budget set as follows. If consumer i wants to consume a specific amount of the public good, then the amount of private good she can consume is diminished by her share r_i of the cost of production of her demanded level of public good. In equilibrium, each consumer i consumes a utility-maximizing bundle of public good and private good within her budget set. In addition, all consumers must choose the same amount of the public

good and the ratios of all consumers must add to 1, so that the costs of public good production are covered. Formally, a *ratio equilibrium* consists of a vector of ratios $\mathbf{r}^* = (r_i^*)_{i \in N}$ and consumption bundles $(x^*, y_i^*)_{i \in N}$ such that

$$\sum_{i \in N} r_i^* = 1,$$

$$x^* \text{ is a solution to } \max_x u_i(x, w_i - r_i^* c(x))$$

for each $i \in N$, and

$$y_i^* + r_i^* c(x^*) = w_i$$

for each $i \in N$.

I illustrate ratio equilibrium in the following example.

Example 3 *Ratio equilibrium in an economy with decreasing returns to scale.*

Consider the 2-consumer public good economy with $c(x) = x^2$, $N = \{1, 2\}$, $w_1 = 4$, $w_2 = 6$, $u_1(x, y_1) = x + y_1$, *and* $u_2(x, y_2) = 3x + y_2$. *Using substitution, we write consumer 1's utility-maximization problem as*

$$\max_x \; x + 4 - r_1^* \left(x^2\right),$$

from which it is easily derived that $x^* = 1/(2r_1^*)$. *From consumer 2's utility-maximization problem we similarly find that* $x^* = 3/(2r_2^*)$. *In equilibrium, these amounts have to be equal, which together with* $r_1^* + r_2^* = 1$ *leads to* $r_1^* = 1/4$ *and* $r_2^* = 3/4$. *Thus, we find that* $x^* = 2$, $y_1^* = 3$, *and* $y_2^* = 3$.

Kaneko (1977) defines ratio equilibrium because "the concept of Lindahl equilibrium has a difficulty in normative meanings" and because "the core never coincides with the Lindahl equilibria". Kaneko refers to Foley (1970) and Milleron (1972), but not to Lindahl (1919) or any of the other papers that I have covered in Section 4, and includes no indication that he is aware that the ratio equilibrium that he defines is a formalization of the ideas presented

in Lindahl (1919). Kaneko (1977) does, however, include a lemma that states that when cost functions are linear, there is an equivalence between ratio equilibria and Lindahl equilibria.[15]

5.2 The axiomatic link between Lindahl (1919) and ratio equilibrium

In van den Nouweland, Tijs, and Wooders (2002), we use the axiomatic method to study ways of determining public good levels and private good consumption by individuals in pure public good economies as described in Section 2. While Lindahl (1919) does not contain an explicit definition of an equilibrium concept, it is easy to capture the ideas expressed in that paper in axioms. Namely, we require that a group of agents takes the shares of total costs paid by other agents as given when making decisions on how much to demand of a public good and how to cover the remaining costs of its production. Based on this requirement, we define two related axioms, consistency and converse consistency, that relate solutions in smaller economies (i.e., those with fewer consumers) to solutions in larger economies. We show that, together with a simple individual rationality axiom, these axioms that capture the cost-share idea expressed in Lindahl's (1919) determine a unique equilibrium concept under the very general conditions of strictly increasing utility for consumption of private and public good and non-decreasing costs for public good production in terms of private good. Interestingly, we find that the unique equilibrium concept that satisfies these axioms based on the ideas in Lindahl (1919) is not Lindahl equilibrium, but ratio equilibrium.

The axiomatic work in van den Nouweland *et al* (2002) strongly suggests that ratio equilibrium is a better fit with Lindahl (1919) than Lindahl equilibrium is, because ratio equilibrium displays characteristics as described in

[15]Kaneko (1977) casually refers to Foley (1970) for validation of this result.

Lindahl (1919), whereas Lindahl equilibrium does not.

5.3 Relations between ratio equilibrium and Lindahl equilibrium

Ito and Kaneko (1981) consider the relationship between Lindahl equilibrium and ratio equilibrium. They do so by looking at similarities and differences between the allocations - levels of public good and each consumer's consumption of private good - that are supported by the two equilibrium concepts. They show that, under a certain condition, every ratio equilibrium allocation can be obtained as a Lindahl equilibrium allocation by varying the re-distribution of the profits from public goods production among the consumers. The condition that they need in their proof is a weak one, namely that when all money in the economy is used to produce public goods, then every individual has a lower utility than she would have when she just consumed her initial endowment of money. Ito and Kaneko (1981) also demonstrate that the possibility of varying the re-distribution of the profits from public goods production among the consumers in a Lindahl equilibrium leads to many more allocations being supported by Lindahl equilibrium than by ratio equilibrium.

Thus, under a mild condition on an economy, the set of ratio equilibrium allocations is a strict subset of the set of Lindahl equilibrium allocations. We see an illustration of this when comparing examples 2 and 3. Both examples look a the same economy, and for this economy we find exactly one ratio equilibrium allocation, but many Lindahl equilibrium allocations. Also, the ratio equilibrium allocation is obtained as a Lindahl equilibrium allocation with distribution rule $d = (1/4, 3/4)$.

Ito and Kaneko (1981) also study what happens to the predictions of the two equilibrium concepts under "cost-linearizing transformations" of units of

measurement of the public goods. The motivation for doing this is two-fold. First, Lindahl equilibrium and ratio equilibrium give the same predictions when public good is measured in terms of its costs, because then a fixed share of the costs is equivalent to a personalized price per unit (of cost). Second, starting from any economy, cost functions can be transformed into linear functions by "measuring the public good in terms of minimal costs required for their production". Thus, it is a desirable property that transforming an economy so that public good is measured in terms of its cost, does not change the equilibrium other than to account for the changed units of measurement. Ito and Kaneko (1981) find that Lindahl equilibrium is not necessarily invariant under such cost-linearizing transformation, whereas ratio equilibrium always is.

I use the economy in examples 2 and 3 to illustrate the cost-linearizing transformation of an economy and that this may change the Lindahl equilibrium allocations.

Example 4 *Cost-linearizing transformation of an economy with decreasing returns to scale.*

Consider the economy in Examples 2 and 3. Measuring the public good in terms of its costs, we denote public good by $X := x^2$. *The cost function and the utility functions of the consumers have to be changed accordingly:* $c(X) = X$, $u_1(X, y_1) = \sqrt{X} + y_1$, *and* $u_2(X, y_2) = 3\sqrt{X} + y_2$.

To find all the Lindahl equilibrium allocations of this economy, note that the profit-maximization problem of the producer only has a non-zero solution if $p_1^* + p_2^* = 1$ *and that the maximum profit of the producer equals 0. Using methods similar to those in Example 1, it is then easily derived that there is a unique Lindahl equilibrium allocation, with* $X^* = 4$, $y_1^* = 3$, *and* $y_2^* = 3$.

Thus, the cost-linearization of the economy results the infinite set of Lindahl equilibrium allocations that we found in Example 2 shrinking to a single allocation. It is easily verified using methods similar to those in Example 3

that this single allocation is also the unique ratio equilibrium allocation of the cost-linearized economy. And, this allocation coincides with the ratio equilibrium allocation that we found in Example 3, because $(x^*)^2 = X^*$.

6 Conclusions

As I have shown, due to some duplicitous use of the word *price*, uneasiness with mathematical formalizations, and carelessness with conditions of constant returns to scale, somewhere between Lindahl (1919) and Milleron (1972), *Lindahl equilibrium* became known as a concept based on *personalized prices*. However, van den Nouweland *et al* (2002) demonstrate that Kaneko's (1977) *ratio equilibrium*, which is based on *personalized shares* of the cost of public good production, embodies better the ideas presented in Lindahl (1919). The fact that, as demonstrated in Ito and Kaneko (1981), the two equilibrium concepts support similar allocations notwithstanding, the two concepts are very different in method. Because personalized prices linearize consumers' payments for public goods, Lindahl equilibrium needs to include profit-maximization requirements and profits need to be somehow re-distributed among consumers. Ratio equilibrium distributes the costs of public good production according to personalized shares and thus automatically results in zero profits.

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A Appendix: The reference graph

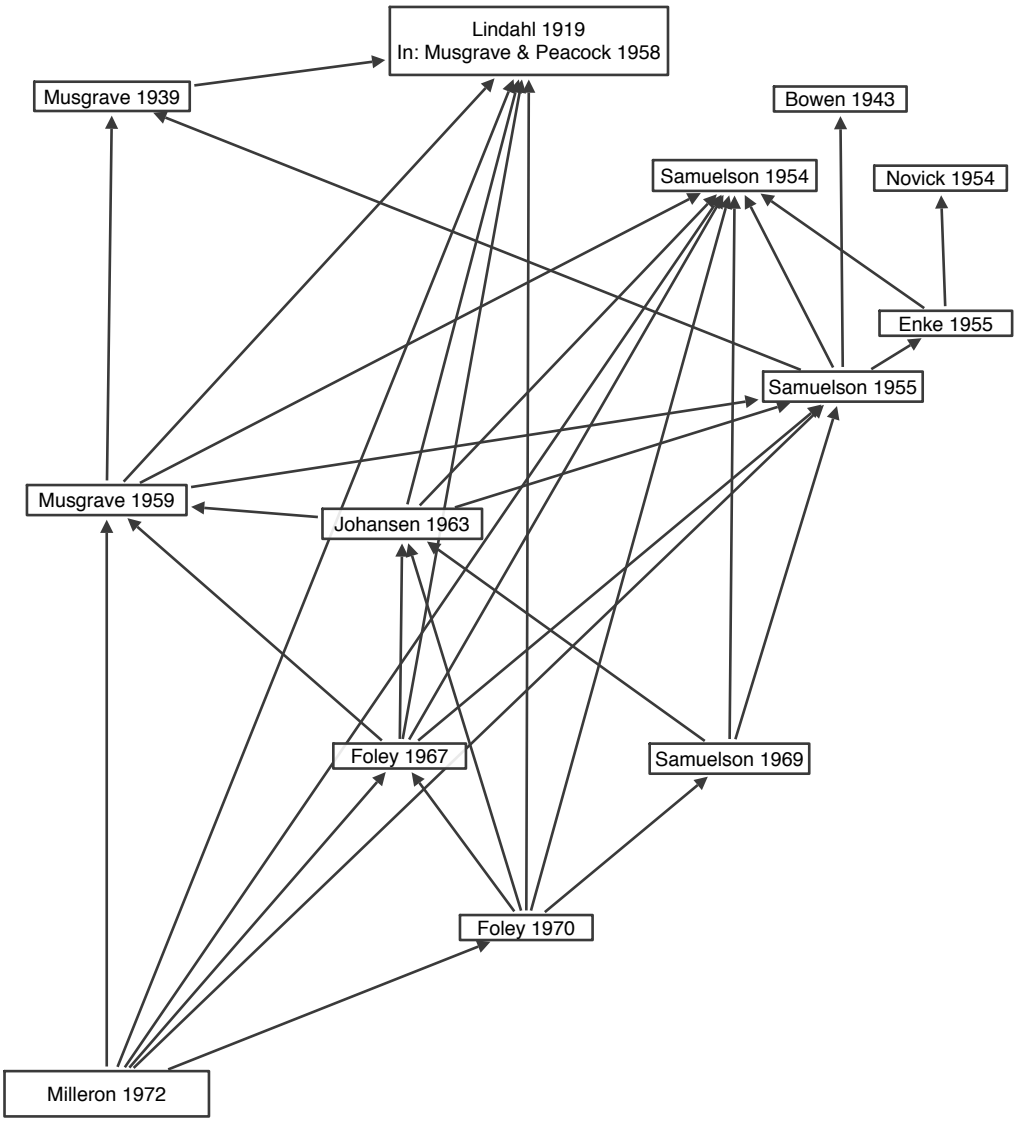

Figure 7: Reference Graph