

Private Information in the BBV Model of Public Goods*

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Abstract

To analyze the private provision of a public good in the presence of private information, we explore the connections between two frameworks: the binary public good model with threshold-uncertainty and the standard continuous model à la Bergstrom *et al.* (1986). Linearity of best responses in others' contributions is key to matching the two frameworks. We identify all utility functions that display this linearity, and we provide conditions ensuring the minimal properties that Bergstrom *et al.* (1986) require of utilities are satisfied. Using techniques developed in the threshold uncertainty framework, we show existence and uniqueness of the Bayes-Nash equilibrium—thus generalizing existing results—and we analyze its comparative statics properties. In particular, under the reasonable assumption that agents' income is stochastic and private information, we complement the full-information crowding-out and redistribution results of Bergstrom *et al.* (1986). If the government taxes agents' income proportionally and redistributes (expected) revenues lump-sum, equilibrium public good provision can increase or decrease, even if the set of contributors is unchanged. Similarly, we show that crowding-out can be one-for-one, less than one-for-one, or more than one-for-one. Finally, we extend our results to a multi-dimensional framework in which agents' unit costs of contributions are also private information.

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1 Introduction

The importance of “On the Private Provision of Public Goods” by Ted Bergstrom, Larry Blume, and Hal Varian (1986)—hereinafter referred to as BBV—to public economics cannot be overstated. In particular, their results on uniqueness, crowding-out and income redistribution “are now staples of public economics and are taught in most graduate courses around the globe.”¹ While the full-information case is well understood, less is known about the voluntary provision of public goods in the BBV setup of continuous contributions and quantity of public good if one posits private information, especially without appealing to mechanism design. As pointed out for instance in Martimort and Moreira (2010), much of the private information literature assumes the existence of an uninformed mediator with full commitment power and then applies the tools of mechanism design. While this assumption is certainly appropriate in some circumstances, Martimort and Moreira (2010) argue it is less so in others.²

Some interesting results appear in this literature. For instance, Gradstein *et al.* (1994) provide examples that show how neutrality with respect to income redistribution can easily fail when private information about willingness to pay is introduced into the BBV model. However, while convincingly establishing that private information has important consequences, the literature has not been able to marry generality and tractability, as BBV and Cornes and Hartley (2007a) do for the full information case. On the one hand, for instance, the tractable (and insightful) private-information, public-good models of Vesterlund (2003) and Andreoni (2006) consider a two- or three-point distribution of uncertainty. Clearly, this limits the kind of information-related questions that these models can answer. On the other hand, Bag and Roy (2011) allow for quite general forms of uncertainty but make no effort in the direction of establishing existence, uniqueness, and comparative statics properties of equilibrium in their analysis of the simultaneous game.³

If one abandons the continuous production function of BBV and focuses instead on a binary public good (a setup such that the public good is either built or not; quantity is not otherwise variable), while retaining continuous contributions, then progress is possible. Earlier works on this subject were limited in focusing primarily on equilibrium existence with two agents, since multiple equilibria typically exist and complicated strategies make deriving comparative statics results forbidding.⁴ The recent contribution of Barbieri and Malueg (2010) adds threshold uncertainty to the discrete model with the result that equilibrium is unique, readily characterized, and easily amenable to comparative statics analysis and applications (see, e.g., Krasteva and Yildirim, 2013, and Barbieri and Malueg, 2014), even for asymmetric environments with

¹See Andreoni and Kanbur (2007, p. 1634). See also the results in Warr (1983) and Cornes and Sandler (1985).

²Martimort and Moreira (2010) include examples such as health, environment, global warming, counter-terrorism, multilateral foreign aid, and lobbying.

³Their focus is on showing the dynamic game outperforms the static one. To do so, they derive an upper bound on simultaneous-game contributions, rather than relying on a detailed characterization of equilibrium.

⁴See e.g. Alboth *et al.* (2001), Menezes *et al.* (2001), Laussel and Palfrey (2003), and Barbieri and Malueg (2008a, 2008b).

many agents. However, Barbieri and Malueg (2010) operate in a binary public-good, subscription-game framework.⁵ While interesting, the applicability of this framework to real world situations is not universal.

As Nitzan and Romano (1990) point out, binary public goods models with threshold uncertainty are closely related to the setup of BBV: the probability of provision can be reinterpreted as a continuous production function. With this interpretation, a natural first question is then, Can the threshold uncertainty approach of Barbieri and Malueg (2010) provide insights into a BBV framework with private information? Indeed, the answer is Yes.

The key simplifying feature in Barbieri and Malueg (2010) is a form of linearity of the first-order optimization conditions. Rather than having to forecast the entire distribution of other agents' strategies, linearity allows a potential contributor to focus only on others' expected contributions, thus considerably reducing the dimensionality of the problem. Our first proposition identifies, for the standard continuous public-good contribution game, which utility functions yield such linear first-order conditions. Our second result then further restricts these utility functions so that they display the properties BBV require of their utility functions: convexity of preferences and strict normality in both the private and the public goods.

Maintaining linear first-order conditions and the BBV properties, we show existence and uniqueness of equilibrium, a substantial generalization of the homologous result of Barbieri and Malueg (2010). We then analyze the comparative statics effects of changing a player's (*ex ante*) distribution of income. Reducing the "riskiness" of a player's distribution of income (in the sense of second-order stochastic dominance) can decrease, increase, or leave constant that player's expected contribution to the public good, depending on whether his contribution strategy is convex, linear, or concave over the "types" that experience the change in income. The overall level of the public good then moves in the same direction as the change in that individual's contribution because even though other players "adjust" in the opposite direction, they do so by less than the player whose income distribution changed. We also explore the consequences of (partial) government provision of the public good that is financed by lump-sum taxes. If taxes are collected only from players who never contribute or only sometimes contribute, then the level of the public good is sure to increase, but by less than the level of government contribution. However, if the tax is collected by players who contribute for all income levels, then the overall supply of the public good remains unchanged—the government's crowding out of private contributions is complete.

We view our results as complementary to BBV's. In a full-information setting with known income levels, they explored the effects of income redistribution involving actual transfers of income across players. In contrast, we consider changes in the income distribution for a particular individual. The BBV redistributions

⁵In their setup, the binary public good is built and paid for if and only if the sum of agents' contributions surpasses a threshold. Otherwise, contributions are refunded. Refunds of insufficient contribution are what distinguish what Admati and Perry (1991) label the *subscription* game from the *contribution* game, which has no refunds.

correspond to an *ex post* greater equalization across players while our “redistributions” correspond to an *ex ante* decrease in riskiness of income for a given player.

The import of these two complementary approaches is on full display when analyzing economies in which income is stochastic and taxation is not lump-sum. For instance, a standard way to model a government intervention that fosters income equality is via the combination of proportional taxation and lump-sum redistribution. In our model, such a scheme would accomplish two changes. First, a lump-sum redistribution of income across agents occurs, from the rich on average to those that are in expectation poor. This across-agent change is well-handled by simply extending the results of BBV at the realized income level. Second, a “riskiness-reducing” redistribution among the possible income realizations of the same agent is effected. Therefore, even if the set of contributors does not ultimately change, contributions to the public good do, in a direction that can be forecasted from the curvature of the contribution strategies.

Similarly and surprisingly, we show in an example that when the government raises taxes through a proportional income tax, then crowding out may be greater than one-for-one: the government’s taxation and contribution yields an overall decrease in the level of the public good. This result obtains because one can decompose the government intervention into two components: first, a public provision of the public good financed through a lump-sum tax set equal to the tax rate times expected income; and second, a reduction in the riskiness of the income distribution because taxes are not actually lump-sum, but proportional. Therefore, even if the first component ends up having no real effect on total provision, the second component matters; by our results on “riskiness-reducing” redistribution, if contribution strategies are convex, then total public good provision decreases.

We conclude our analysis by considering an example of multi-dimensional private information: the case in which, in addition to income, agents’ unit costs of contributions are also private information. The possibility of different unit costs of contributions is well analyzed, with full information, by Cornes and Hartley (2007a). We complement their analysis of changes in levels of unit costs by determining the effects of changes in the distribution of unit costs.

The papers using frameworks most similar to ours appear in the baseline simultaneous models used in the theoretical literature on sequential giving or leadership in teams. The most important difference with our work is in focus. All these papers analyze the effects of dynamics in contribution, as in Varian (1994). None deals with issues of inequality and redistribution or taxation and crowding-out. Moreover, there exist important technical differences with our approach as well. We consider private information about a private-value characteristic—income—while the majority of these papers deal with a common-value characteristic, e.g., the quality of the charity or the total factor productivity of the production function of the public good. This is the case for Hermalin (1998), Vesterlund (2003), Kobayashi and Suehiro (2005), Andreoni (2006),

Komai, Stegeman, and Hermalin (2007), and Komai and Stegeman (2010). The fact that the information is about a common-value characteristic is important because it creates a signaling motivation for leadership contributions and information acquisition. This leads to the vast majority of contributors making their contribution decisions without facing private information: all contributors but the leader observe the same action by the leader and they all draw from it the same inference about the quality of the public good. Our framework is different since there is no signaling motivation for giving and every agent has private information. This is also true of Bag and Roy (2008) and (2011), who consider a private-value private-information model, as we do. Their focus is however on the dynamics of contributions, rather than on performing an in-depth analysis of the simultaneous game. Therefore, they establish an upper bound for contributions in the simultaneous game, which is sufficient to show conditions under which dynamic provision outperforms static provision. But they sidestep issues of equilibrium uniqueness, which prove tricky in their framework. Indeed, even for the two-player, two-type case, they say that “. . . in general it is not possible to guarantee uniqueness of equilibrium contributions or even expected total contribution in the one-shot game” (Bag and Roy, 2008, p. 66).

There exists a small but growing empirical literature on the effects on income inequality on the private provision of public goods. Payne and Smith (forthcoming) provides both a recent contribution and a review of existing results, which point to the possibility that increased inequality may increase, decrease, or have no statistically discernible effect on voluntary contributions. Our results may help rationalize this variety. Most importantly, our results provide a theoretical baseline for when the agents’ motivation for giving is the “enlightened self-interest” of BBV; empirical departures from this baseline can be taken as evidence of the importance of additional motivations for giving—signaling or warm-glow, for instance.

The rest of the paper is organized as follows. The next section explores the relationship between the binary public-good model with threshold uncertainty and the standard continuous case for full-information environments. This allows us to gather intuition on how to proceed when information asymmetries are present. Section 3 describes our private information model and presents our matching results. Section 4 analyzes equilibrium and the curvature of equilibrium contribution functions. Through several examples, in Section 5 we illustrate our results and their implications for crowding out and redistribution; we also analyze the case of different unit costs of contributions. Section 6 concludes. The Appendix contains most proofs and a numerical simulation for a utility function that does not satisfy our linearity condition; the results we obtain there agree with our analytical findings.

2 The full-information environment

At the moment in which a joint endeavor is first undertaken, uncertainty about what the actual cost will turn out to be at completion is almost a fact of life.⁶ It is therefore not surprising that such uncertainty occupies an important place in economic theory. Of particular relevance is the seminal contribution of Nitzan and Romano (1990) on binary public goods. The standard modeling of binary public goods has the good provided if and only if the sum of agents' contributions exceeds a set threshold, as in Palfrey and Rosenthal (1984, 1988), for instance. Referring to a firm engaged in a court case that will set precedent for others, Nitzan and Romano (1990) noted that "it is probably more realistic to assume that... the probability of winning was an increasing function of the resources devoted... rather than Palfrey and Rosenthal's assumption that winning required some known fixed expenditure."⁷ Thus they were motivated to introduce uncertainty in the cost threshold.

Beyond arguing that cost uncertainty is realistic, Nitzan and Romano (1990) demonstrate that threshold uncertainty introduces important differences in equilibrium prediction as compared with the fixed-threshold case: with sufficient uncertainty, equilibrium is unique and inefficient. As Nitzan and Romano (1990) note, these equilibrium predictions are familiar from the standard continuous case of BBV.

The similarity is not coincidental: one can readily map the setup of Nitzan and Romano (1990) into the one of BBV through a simple reinterpretation of the individual utility function of agent i . Nitzan and Romano (1990) describe the utility function in their equation (5) on page 362, and we report it here:

$$(\Delta_i - c_i)F\left(\sum c_j\right).$$

If one interprets F , the cumulative distribution of the cost threshold, as the quantity of public good produced, Δ_i as income, and c_i as agent i 's contribution to the public good, then one gets a special case of the setup of BBV, in which the utility function is (Private good) \times (Public good).

In the rest of this paper we explore what additional complications to the above described procedure are introduced by private information and what utility functions, beyond (Private good) \times (Public good), allow this procedure to go through.

⁶E.g., Krasteva and Yildirim, (2013, p. 16): "For instance, the exact cost of a construction project may be the result of a procurement auction; the price of a high-tech equipment needed for a radio program may depend on fluctuating supply conditions; and the minimum number of ticket sales needed for a concert may be uncertain due to the rival venues."

⁷See Nitzan and Romano (1990, p. 358).

3 The model

We generally follow the notation of BBV. We study the problem of n players who simultaneously contribute to the funding of a public good. Player i has preferences represented by the utility function $u_i(x_i, \tilde{G})$, where $x_i \geq 0$ is the realized quantity of a private good and \tilde{G} is the total realized quantity of a pure public good, $i = 1, \dots, n$.⁸ Desirability of the goods and player i 's budget constraint imply $x_i + \tilde{g}_i = w_i$, where $\tilde{g}_i \geq 0$ is player i 's contribution to the public good and w_i is i 's exogenous income. The production function for the public good is simply $\tilde{G} = \tilde{g}_1 + \dots + \tilde{g}_n$.

Departing from BBV, we model this provision game as one of private information, where player i 's income is independently drawn from the cumulative distribution F_i . We let \underline{w}_i and \bar{w}_i denote the infimum and supremum of the support of F_i . A player knows his own realized income but no one else's. However, distributions are commonly known.

We begin our analysis by looking for a Bayes-Nash equilibrium $\{g_1(\cdot), \dots, g_n(\cdot)\}$ to the simultaneous voluntary contribution game, where $g_i(w_i)$ is player i 's contribution when his income is w_i . Provisionally, we derive best responses through the first-order conditions (FOC) for an interior $g_i(w_i)$ (i.e., $0 < g_i(w_i) < w_i$). The condition reads

$$\mathbb{E} \left[\frac{\partial u_i}{\partial x_i} \left(w_i - g_i(w_i), g_i(w_i) + \sum_{j \neq i} g_j(w_j) \right) - \frac{\partial u_i}{\partial \tilde{G}} \left(w_i - g_i(w_i), g_i(w_i) + \sum_{j \neq i} g_j(w_j) \right) \middle| w_i \right] = 0, \quad (1)$$

where the expectation is taken with respect to $\{w_j\}_{j \neq i}$ given w_i . Through Proposition 2 we will ensure that u_i 's curvature properties make equation (1) necessary and sufficient to identify best responses.

A significant difficulty arises in dealing with condition (1). In general, agents must forecast the *entire distribution* of every other agent's contribution and then calculate the convolution of these distributions. The setup of Barbieri and Malueg (2010) is especially convenient because agents need only forecast the *average* contribution of other agents. As in that paper, if it is possible to transport the expectation operator inside the marginal utilities in (1), then one can greatly simplify the problem. Thus, if

$$\frac{\partial u_i}{\partial x_i} \left(w_i - g_i(w_i), g_i(w_i) + \sum_{j \neq i} g_j(w_j) \right) - \frac{\partial u_i}{\partial \tilde{G}} \left(w_i - g_i(w_i), g_i(w_i) + \sum_{j \neq i} g_j(w_j) \right) \quad (2)$$

is "linear in others' contributions"—that is, if it can be reformulated as

$$h_i(w_i - g_i(w_i)) + k_i(w_i - g_i(w_i)) \times \left(g_i(w_i) + \sum_{j \neq i} g_j(w_j) \right), \quad (3)$$

⁸Because u_i is the subject of our analysis, we leave a discussion of its curvature properties to Proposition 2.

where h_i and k_i are continuous real-valued univariate functions—then one obtains the desired simplification.

Essentially, the equivalence of conditions (2) and (3) obtains for the solution $u_i(\cdot, \cdot)$ of the partial differential equation

$$\frac{\partial u_i}{\partial x_i}(x_i, \tilde{G}) - \frac{\partial u_i}{\partial \tilde{G}}(x_i, \tilde{G}) = h_i(x_i) + k_i(x_i) \times \tilde{G}. \quad (4)$$

The following proposition (proven in the Appendix) derives the set of utility functions that display the required linearity.

Proposition 1 (Equivalence). *The solution to (4) is*

$$u_i(x_i, \tilde{G}) = \int_0^{x_i} [h_i(s) + k_i(s)(x_i + \tilde{G} - s)] ds + v_i(x_i + \tilde{G}), \quad (5)$$

where v_i is an arbitrary real-valued univariate function.

The import of Proposition 1 is that the relevant first-order condition for utility function $u_i(x, G)$ will be linear in G if and only if it has the form in (5) for some univariate functions h_i , k_i , and v_i . It is convenient to restate the utility function in (5) as

$$\begin{aligned} u_i(x_i, \tilde{G}) &= \int_0^{x_i} [h_i(s) + k_i(s)(x_i - s)] ds + \tilde{G} \int_0^{x_i} k_i(s) ds + v_i(x_i + \tilde{G}) && \text{(by (5))} \\ &= \underbrace{\int_0^{x_i} [h_i(s) + K_i(s)] ds}_{\equiv H_i(x_i)} + \tilde{G} \underbrace{\int_0^{x_i} k_i(s) ds}_{\equiv K_i(x_i)} + v_i(x_i + \tilde{G}), && (6) \end{aligned}$$

where the second equality uses integration by parts. Given any functions H_i and K_i , suitable functions h_i and k_i can be found to generate the associated utility function. Thus, rather than working with h_i and k_i , one may specify H_i and K_i independently. Judicious choices of the functions H_i , K_i , and v_i yield familiar utility functions, including $u_i(x_i, \tilde{G}) = x_i^\alpha \tilde{G}$, $u_i(x_i, \tilde{G}) = \alpha_1 x_i - \alpha_2 x_i^2 + \beta_1 \tilde{G} - \beta_2 \tilde{G}^2 + \gamma x_i \tilde{G}$, $u_i(x_i, \tilde{G}) = x_i^\alpha + \tilde{G}$, and $u_i(x_i, \tilde{G}) = \alpha x_i + \beta \tilde{G}$, where $\alpha > 0$ and $\beta > 0$. Linear combinations of these utility functions are also admissible solutions.⁹

With the restatement of utility in (6), the FOC (1) becomes

$$0 = \frac{dE \left[u_i(w_i - g_i(w_i), g_i(w_i) + \sum_{j \neq i} g_j(w_j)) \mid w_i \right]}{dg_i(w_i)},$$

⁹To obtain $u_i(x_i, \tilde{G}) = \alpha_1 x_i - \alpha_2 x_i^2 + \beta_1 \tilde{G} - \beta_2 \tilde{G}^2 + \gamma x_i \tilde{G}$, we take $H_i(x) = \alpha_1 x + (\beta_2 - \alpha_2)x^2$, $K_i(x) = \beta_1 + (\gamma + 2\beta_2)x$, and $v_i(x + \tilde{G}) = -\beta_2(x + \tilde{G})^2$. Because our framework accommodates quadratic utility, the functional form in (5) encompasses a second-order Taylor expansion of any utility function. Thus, our model, in the words of Angeletos and Pavan (2007, p. 1109), "... might also be viewed as a second-order approximation of a broader class of concave economies."

or

$$0 = -H'_i(w_i - g_i(w_i)) - K'_i(w_i - g_i(w_i)) \times (G_{-i} + g_i(w_i)) + K_i(w_i - g_i(w_i)), \quad (7)$$

where we define

$$G_j = E[g_j(w_j)] \quad \forall j \quad \text{and} \quad G_{-i} = \sum_{j \neq i} G_j \quad \forall i;$$

that is, G_j is the expected contribution of player j . Note that in (7) the function v_i does not appear because, using the budget constraint, the argument $x_i + \tilde{G} = \sum_{j \neq i} \tilde{g}_j + w_i$ is independent of $g_i(w_i)$.

We now proceed with the functional form in (6) and we identify properties of H_i and K_i in light of two objectives. First, we want our reliance on the FOC to be justified. Second, we want u_i in (6) to satisfy the mild assumption that BBV impose on preferences. For a clean statement of our sufficient conditions, in the next proposition we set the arbitrary function v_i to a constant \bar{v} ; without further loss of generality we take $\bar{v} = 0$.

Proposition 2 (Properties). *Suppose the u_i have the form in (6) with $v_i \equiv 0$. If the following conditions are met:*

1. $K_i > 0$ for all $x_i > 0$;
2. $H'_i \geq 0$ and $K'_i > 0$ for all $x_i > 0$; and
3. $H''_i \leq 0$ and $K''_i \leq 0$;

then the following properties hold:

- i. Both goods are strictly desirable;
- ii. The FOC characterizes interior best responses;
- iii. When $x_i > 0$ and $g_i > 0$ they are strictly normal; and
- iv. Preferences are strictly convex.

The sufficient conditions in Proposition 2 imply that the marginal utility of x_i is decreasing in x_i , but it is increasing in \tilde{G} .

While Proposition 2 yields a match with BBV, the assumptions are stronger than necessary. What matters for the rest of the analysis are the conclusions of Proposition 2, and these do not require setting v_i to zero.¹⁰

¹⁰One may easily verify that the quadratic utility in footnote 9, which requires $v_i \neq 0$, satisfies the conclusions of Proposition 2 for a carefully restricted domain. Similarly, the Cobb-Douglas $x^\alpha \tilde{G}$, which has $v_i = 0$ but does not satisfy condition 3 if $\alpha > 1$, still exhibits properties i-iv.

4 Equilibrium and comparative statics

We now proceed to characterize equilibrium with the utility function given by

$$u_i(x_i, \tilde{G}) = H_i(x_i) + K_i(x_i)\tilde{G} + v_i(x_i + \tilde{G}), \quad (8)$$

where H_i and K_i satisfy the following three assumptions:

A1 Properties i–iv in Proposition 2 hold;

A2 $\lim_{x \rightarrow 0} \frac{\frac{\partial u_i}{\partial x_i}(x, \tilde{G})}{\frac{\partial u_i}{\partial G}(x, \tilde{G})} > 1$, $\forall \tilde{G}$ and $\forall i$; and

A3 For some i and some $\tilde{w} < \bar{w}_i$, it is the case that $\frac{\frac{\partial u_i}{\partial x_i}(\tilde{w}, 0)}{\frac{\partial u_i}{\partial G}(\tilde{w}, 0)} < 1$.

Assumption A2 states that the marginal rate of substitution (MRS) is strictly larger than 1 in a neighborhood of $x = 0$, so that no agent finds it optimal to contribute all of her income to the public good. Assumption A3 similarly states that the MRS, when evaluated $(w_i, 0)$, is strictly less than 1 with strictly positive probability for at least one agent, so the no-contribution strategy profile is never an equilibrium. For the rest of this paper, we maintain assumptions A1–A3.

First, we establish existence and uniqueness of equilibrium. Then we derive the effects of changing a player's distribution of income. In particular, we show that decreasing the riskiness of a player's income distribution may increase or decrease total expected contributions, depending on the shape of the player's contribution strategy.

We begin by characterizing best responses. Using (7), the FOC that characterizes interior best responses can thus be rewritten as

$$\underbrace{\frac{K_i(w_i - g_i(w_i)) - H'_i(w_i - g_i(w_i))}{K'_i(w_i - g_i(w_i))}}_{\equiv Q_i(w_i - g_i(w_i))} - g_i(w_i) = G_{-i}. \quad (9)$$

Implicitly differentiating (9) with respect to w_i , we obtain

$$g'_i = Q'_i(1 - g'_i), \quad (10)$$

which, by Assumption A1, implies $Q'_i > 0$ wherever g_i is positive. Therefore, the left-hand side of (9) is strictly decreasing in $g_i(w_i)$ under Assumption A1. Thus, equation (9) defines $g_i(w_i)$ as a function of G_{-i} in two steps. First, if (9) does not admit a positive solution for $g_i(\bar{w}_i)$, then $g_i(w_i) = 0 \forall w_i$. In this case,

the auxiliary function $\hat{w}_i(G_{-i})$ can be set at \bar{w}_i . Second, if $g_i(\bar{w}_i) > 0$, then strict normality implies that, starting at \bar{w}_i and progressively reducing w_i , (9) defines $g_i(w_i)$ as a function of G_{-i} while $g_i(w_i)$ remains positive. If in the course of this procedure $g_i(w_i)$ becomes zero, we can set $g_i(w_i) = 0$ for any $w_i \leq \hat{w}_i(G_{-i})$, where $\hat{w}_i(G_{-i})$ solves

$$Q_i(\hat{w}_i(G_{-i})) = G_{-i}. \quad (11)$$

If in the course of this procedure $g_i(w_i)$ never becomes zero, then we can set $\hat{w}_i(G_{-i})$ to any arbitrary value strictly less than \underline{w}_i . We label the contribution function thus defined as $b_i(w_i, G_{-i})$, so that we obtain player i 's expected contribution:

$$G_i = \int_{\hat{w}_i(G_{-i})}^{\bar{w}_i} b_i(w_i, G_{-i}) dF_i(w_i). \quad (12)$$

The following proposition, with proof in the Appendix, contains our main uniqueness result.¹¹

Proposition 3 (Existence and uniqueness). *Suppose the u_i have the form in (8) and Assumptions A1–A3 are satisfied. Then there exists a unique equilibrium.*

We now turn attention to comparative statics. First, we show how agents' expected contributions respond to an exogenous contribution, Δ , by another party. Given this exogenous contribution, let $g_i^*(w_i, \Delta)$ denote player i 's equilibrium contribution when i 's income realization is w_i . Similarly, let $G_i^*(\Delta)$ be i 's expected equilibrium contribution in the n -player game, and let $G^*(\Delta)$ denote the associated total expected contributions of the n players. Finally, note that the income level of player i that must be exceeded for equilibrium contributions to be strictly positive is $\hat{w}_i(G^*(\Delta) - G_i^*(\Delta) + \Delta)$. Lemma 1 below establishes that in response to an increase in exogenous contributions, all agents reduce their expected contributions, but not sufficiently to reduce the total expected provision. Moreover, the reduction in contributions occurs not just in expectation, but for each income level, both at the extensive and intensive margins.

Lemma 1. *If $G_i^*(\Delta) > 0$, then $-1 < \frac{dG_i^*(\Delta)}{d\Delta} < 0$ and $-1 < \frac{dG^*(\Delta)}{d\Delta} < 0$. Moreover, $\frac{d\hat{w}_i(G^*(\Delta) - G_i^*(\Delta) + \Delta)}{d\Delta} > 0$ if $\hat{w}_i(G^*(\Delta) - G_i^*(\Delta) + \Delta) \in (\underline{w}_i, \bar{w}_i)$, and $\frac{\partial g_i^*(w_i, \Delta)}{\partial \Delta} \leq 0$, with strict inequality for all w_i where $g_i^* > 0$.*

Second, we determine the concavity or convexity of the contribution function: implicitly differentiating (9) twice yields

$$g_i''(1 + Q_i') = Q_i''(1 - g_i')^2, \quad (13)$$

¹¹Proposition 3 substantially generalizes the results of Barbieri and Malueg (2010) since the functions b_i do not need to be piecewise-linear in w_i or G_{-i} , as arise in Barbieri and Malueg (2010). All examples in Gradstein *et al.* (1994) are piecewise linear in w_i and G_{-i} .

so we see that the signs of g_i'' and Q_i'' are the same.¹²

Consider now distributions F_1, F_2, \dots, F_n for players' values and denote the unique expected equilibrium contributions as $G_1^*, G_2^*, \dots, G_n^*$. Next take the distributions $\hat{F}_1, F_2, \dots, F_n$ and let the expected equilibrium contributions in this case be $\hat{G}_1^*, \hat{G}_2^*, \dots, \hat{G}_n^*$. For ease of exposition, assume that in both equilibria player 1 contributes for all income levels and assume that both G_{-1}^* and \hat{G}_{-1}^* are strictly positive. The following proposition derives effects of changing player 1's income distribution.

Proposition 4 (Stochastic dominance). *Fix the distributions F_2, \dots, F_n . Consider two distributions for agent 1's values, F_1 and \hat{F}_1 . Assume that in both equilibria player 1 contributes for all income levels and assume that both G_{-1}^* and \hat{G}_{-1}^* are strictly positive.*

- a. *Suppose F_1 strictly first-order stochastically dominates \hat{F}_1 . Then $\hat{G}_1^* < G_1^*$, $\hat{G}_{-1}^* > G_{-1}^*$, and $\hat{G}^* < G^*$.*
- b. *Suppose \hat{F}_1 strictly second-order stochastically dominates F_1 and Q_1 is strictly convex. Then $\hat{G}_1^* < G_1^*$, $\hat{G}_{-1}^* > G_{-1}^*$, and $\hat{G}^* < G^*$.*
- c. *Suppose \hat{F}_1 strictly second-order stochastically dominates F_1 and Q_1 is strictly concave. Then $\hat{G}_1^* > G_1^*$, $\hat{G}_{-1}^* < G_{-1}^*$, and $\hat{G}^* > G^*$.*

Proof. Lemma 1 allows a reaction-function analysis of equilibrium contributions, even when the number of players is larger than 2. We separate out player 1 from the aggregate of all other players, and represent expected contributions on a Cartesian plane. Using (9) we define the following (expected-contribution) reaction function for player 1,

$$G_1 = R_1(G_{-1} | F_1) \equiv \int_{\hat{w}_1(G_{-1})}^{\bar{w}_1} b_1(w_1, G_{-1}) dF_1(w_1), \quad (14)$$

for given G_{-1} , where b_1 was defined following (11). Observe that the function R_1 is continuous and strictly decreasing with slope less than 1 in absolute value. Similarly, we can consider the game played by all players other than player 1 as defining an aggregate reaction function for all other players, given some exogenously fixed level for G_1 . The sum of all solutions G_2, \dots, G_n then can be written as $R_{-1}(G_1)$ and, in equilibrium $G_{-1}^* = R_{-1}(G_1^*)$. By Lemma 1, with G_1 assuming the role of Δ , we also see R_{-1} is strictly decreasing, with slope less than 1 in absolute value.

Part a. Let $R_1(G_{-1} | F_1)$ and $R_1(G_{-1} | \hat{F}_1)$ be defined as above. To establish part a, observe by (14) and Proposition 2, because b_1 is increasing in w_1 , changing player 1's distribution from F_1 to \hat{F}_1 shifts

¹²The example of Section 5.1 suggests that there is no easily interpretable condition on u_i that determines whether Q_i is concave or convex. As an empirical matter, the procedure and difficulties involved in determining the curvature of $g_i(w_i)$ are discussed by McClelland and Brooks (2004), for instance. And since the signs of g_i'' and Q_i'' are the same, the curvature of Q_i can be determined empirically, in principle.

player 1's reaction function leftward, as shown in Figure 1. Therefore, the equilibrium values of the expected contributions (G_1, G_{-1}) for the profile $(\hat{F}_1, F_2, \dots, F_n)$ will be on the function R_{-1} , to the northwest of the equilibrium expected contributions for the profile (F_1, F_2, \dots, F_n) (the key feature of the reaction functions is that their slopes lie between -1 and 0). This reasoning establishes $\hat{G}_1^* < G_1^*$ and $\hat{G}_{-1}^* > G_{-1}^*$ in part a, and it is illustrated in Figure 1. The fact that $\hat{G}^* < G^*$ follows because, by Lemma 1, the slope of R_{-1} lies in $(-1, 0)$, again as illustrated in Figure 1, so $(\hat{G}_1^*, \hat{G}_{-1}^*)$ lies below the line with slope -1 through the point (G_1^*, G_{-1}^*) .

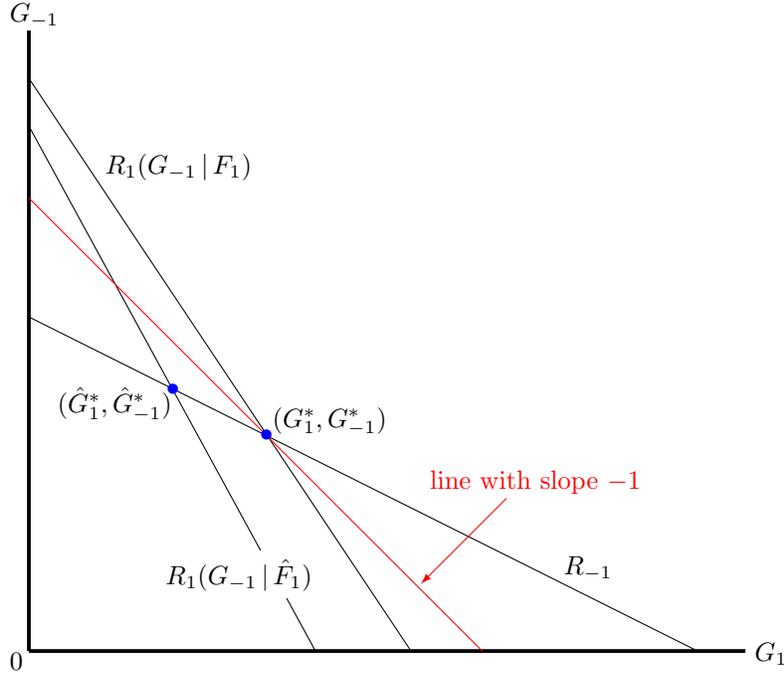


Figure 1: F_1 FOSD \hat{F}_1 or \hat{F}_1 SOSD F_1 , assuming Q_1 is convex

Part b. It is enough to note that, if Q_1 is strictly convex, then b_1 is strictly convex in w_1 . Since b_1 is the integrand in equation (14), the fact that \hat{F}_1 strictly second-order stochastically dominates F_1 a leftward shift to R_1 as in part a. Therefore, the same conclusion results.

Part c. If Q_1 is strictly concave, then b_i is strictly concave. Changing player 1's distribution of income from F_1 to \hat{F}_1 shifts R_1 rightward, and the result follows from reversing the previous reasoning. \square

Of course, as usual, Proposition 4 applies with weak inequalities if the convexity of Q is not strict or if the dominance shifts are only weak. Moreover, if player 1 does not contribute for all income levels, all assumptions of the proposition must be adapted so that they apply to the range of income levels for which $g_1^* > 0$ in the original distribution, leaving everything else unchanged. Informally, when one player's income distribution shifts to the right, his expected contribution increases; this is partially offset by other players

reducing their contributions, but overall the effect is that expected contributions increase. For a player with a convex contribution strategy, a reduction in the riskiness of his income distribution will reduce his expected contribution; this is partially offset by other players increasing their contributions, but overall the effect is that expected contributions decrease. The effect is reversed for a player whose contribution strategy is concave in the region where it is strictly positive; if this player is sure to contribute at all possible incomes, then reducing the riskiness of his income distribution leads him to increase his contributions, with total contributions also increasing. A neutrality result is found when the contribution strategy is linear where it is positive: if such a player contributes at all possible income levels, then a reduction in the riskiness of his income distribution has no effect on his expected contribution.

It must be pointed out that the results in Proposition 4 about expected contributions do not simply obtain because of the averaging of a, say, concave contribution function that remains unchanged. On the contrary, the following Corollary demonstrates that the same patterns of changes obtain for the equilibrium contribution functions $g_i^*(w_i)$ and $\hat{g}_i^*(w_i)$, for each income realization w_i . For clarity of exposition, we think of G_i^* as describing the initial situation and of \hat{G}_i^* as describing the final equilibrium. We state the Corollary and conduct the proof only for the changes described in part c of Proposition 4 to align the direction of changes with the previous description; however, all other cases follow similarly.

Corollary 1. *Under the assumptions of part c of Proposition 4, $\hat{g}_1^*(w_1) > g_1^*(w_1) \forall w_1$. Moreover, for $i = 2, \dots, n$, $G_i^* \geq \hat{G}_i^*$; and $G_i^* > \hat{G}_i^* \implies g_i^*(w_i) > \hat{g}_i^*(w_i)$ for all w_i levels such that $g_i^*(w_i) > 0$.*

Proof. Since $\hat{G}_{-1}^* < G_{-1}^*$, it follows by (9) that $\hat{g}_1^*(w_1) > g_1^*(w_1)$ for all w_1 , in addition to $\hat{G}_1^* > G_1^*$ as already determined in Proposition 4. Consider now players $2, \dots, n$ and apply Lemma 1 with the expected contribution of agent 1 taking the place of Δ . Immediately, we obtain that $G_i^* \geq \hat{G}_i^*$, $\forall i > 1$. To conclude the proof, let $G_i^* > \hat{G}_i^*$ for some $i > 1$ and consider any w_i such that $g_i^*(w_i) > 0$. We have

$$\begin{aligned} g_i^*(w_i) - \hat{g}_i^*(w_i) &= \int_{\hat{G}_1^*}^{G_1^*} \frac{\partial g_i(w_i, \Delta)}{\partial \Delta} d\Delta \\ &= - \int_{G_1^*}^{\hat{G}_1^*} \frac{\partial g_i(w_i, \Delta)}{\partial \Delta} d\Delta \\ &> 0, \end{aligned}$$

where the strict inequality follows by Lemma 1. To see this, note that $\frac{\partial g_i(w_i, \Delta)}{\partial \Delta} < 0$ over an interval that takes strictly positive probability, i.e., for Δ sufficiently close to G_1^* . Moreover, if as Δ increases from G_1^* to \hat{G}_1^* the contribution for some income level becomes zero, that income-level contribution remains zero for larger values of Δ as well, because of the properties of \hat{w} established in Lemma 1. \square

The results in Proposition 4 have an interesting parallel with those in Theorem 5 in BBV. In particular, part (iii) of BBV’s Theorem 5 states that “Equalizing wealth redistributions will never increase the voluntary equilibrium supply of the public good,” and we find that if Q_1 is convex, for example, then reducing the riskiness of player 1’s income distribution does not increase the expected supply of the public good. And part (iv) of BBV’s Theorem 5 states that “Equalizing wealth redistributions among current non-contributors or among current contributors will leave the equilibrium supply unchanged.” In our setting, we find, even for a player who contributes at all possible income levels, reducing the riskiness of a player’s income distribution may increase or decrease his expected contribution (and so, too, the overall expected level of the public good) according to whether his contribution strategy is concave or convex.

Applying Proposition 4 to players one at a time, we obtain the following corollary.

Corollary 2. *If all player’s contribution strategies are convex, then reducing the riskiness of each player’s distribution of income reduces the overall expected level of the public good. If all players’ contribution strategies are concave and for all relevant income levels each player contributes to the public good, then reducing the riskiness of each player’s distribution of income increases the overall expected level of the public good.*

In the literature on taxation, it is common to assume income to be stochastic (see, e.g., Kanbur, 1981), a feature accounted for by our framework. In the political-economy literature on inequality, a standard way to model inequality-reducing income redistribution is via a combination of proportional taxation and lump-sum redistribution (see, e.g., Persson and Tabellini, 1994). In our model, such a scheme would accomplish two changes. First, a lump-sum redistribution of income across agents occurs, from the rich on average to those that are in expectation poor. This across-agent change is well-handled by simply extending the results of BBV at the realized income level.¹³ Second, a “riskiness-reducing” redistribution among the possible income realizations of the same agent is effected.¹⁴ Here, Proposition 4 and Corollaries 1 and 2 apply, so that interim and expected contributions change as described above, in a way that goes beyond what a simple extension of BBV’s results may suggest.

We conclude this section with an analysis of crowding-out, analyzing the effects of taxation and government provision on voluntary provision of a public good. Here we favor clarity and simplicity over realism by assuming taxes are lump-sum and can be paid by each individual for all possible income realizations. The government then uses these tax revenues to contribute an equal amount to the public good. Our modeling assumptions are implemented through the following timing:

¹³The procedure and results are completely analogous to the upcoming Proposition 5.

¹⁴The risk-reducing effects of taxation have been formally analyzed at least since Domar and Musgrave (1944). The extensive literature stemming from this seminal paper deals with portfolio or occupational choice, for instance. However, the risk-reducing effects of taxation on the voluntary provision of public goods à la BBV have not been studied. (There is a large literature on the tax-price of deductible charitable contributions; this is not at all the issue here.)

1. The government announces a lump-sum taxation scheme (t_1, \dots, t_n) , where t_i is the tax levied on agent i , and contributes $\sum t_j$ to the public good;
2. Each agent receives his income realization and pays taxes accordingly;
3. Each agent, simultaneously and independently, voluntarily contributes to the public good; and
4. The total public good provision is the sum of the government's contribution and the agents' contributions.

We now obtain the following counterpart of Theorem 6 in BBV.

Proposition 5 (Crowding out). *Suppose that starting from an initial position where consumers supply a public good voluntarily, the government supplies some amount Δ of the public good, which it pays for through lump-sum taxes.*

1. *If the taxes are collected from agents that never contribute to the public good, then expected voluntary contributions decrease, but by less than Δ . Therefore, the equilibrium total supply of the public good increases.*
2. *If the taxes are collected from agents that contribute to the public good, then expected voluntary contributions decrease. The decrease is less than Δ if for some contributor $\hat{w}_i > \underline{w}_i$ in the after-tax equilibrium. The decrease in total contributions is exactly Δ if for all contributors $\hat{w}_i \leq \underline{w}_i$ in the after-tax equilibrium. In this last case, the equilibrium total supply of the public good stays the same.*

While we leave the proof to the Appendix, it is important to remark here that taxes must be *lump-sum*, i.e., independent of the actual realization of w_i . Otherwise, one needs to take into account the distributional consequences of taxation. Indeed, in the next section we present an example that shows that, when equilibrium strategies are strictly convex and taxation is proportional, crowding-out may be more than one-for-one.

5 Examples, extensions, and implications

In this section, we provide a simple family of examples for which the contribution strategies can be convex or concave where they are strictly positive. The examples highlight how the techniques we have developed allow us to extend the conclusions of Barbieri and Malueg (2010) for both applications and theoretical insights. Suppose there are two players, 1 and 2, with utility functions as in (8), with the additional assumptions that

$v_i \equiv 0$, $H_i(x) = x^b$, and $K_i(x) = x^a$, with $a \in (0, 1]$ and $b \in [0, 1]$. Then preferences satisfy the assumptions of Proposition 2. We have

$$Q_i(x) = \frac{K_i(x) - H_i'(x)}{K'(x)} = \frac{x - bx^{b-a}}{a}$$

and

$$Q_i''(x) = -\frac{(a-b)b(1+a-b)x^{b-a-2}}{a}.$$

Therefore, where it is positive, the equilibrium strategy will be concave if $a > b$ and convex if $a < b$.

5.1 Comparative statics, taxation, government provision, and crowding-out

Here we suppose there are symmetric players and only three possible income levels, $w_L = 1$, $w_M = 1.5$, and $w_H = 2$. The associated probabilities are symmetric about the expected income of 1.5: $\Pr(w_i = w_M) = p_M$ and $\Pr(w_i = w_L) = \Pr(w_i = w_H) = (1 - p_M)/2$. Thus, reductions in p_M yield riskier distributions of income. For simplicity, the examples are such that players contribute at all possible income levels.

Table 1: Voluntary contributions, without and with taxation to finance government provision

	$a = \frac{3}{4}, b = \frac{1}{4}$ (g is concave)	$a = b = \frac{1}{2}$ (g is linear)	$a = \frac{1}{4}, b = \frac{3}{4}$ (g is convex)
$p_M = 1$	$G_i = 0.5$	$G_i = 0.5$	$G_i = 0.5$
$p_M = \frac{1}{3}$	$G_i = 0.499026$	$G_i = 0.5$	$G_i = 0.501148$
$p_M = 0$	$G_i = 0.498538$	$G_i = 0.5$	$G_i = 0.501721$
$p_M = \frac{1}{3}, \tau = \frac{0.1}{1.5}, l = 0.1$	$G_i = 0.499154$	$G_i = 0.5$	$G_i = 0.500998$
$p_M = \frac{1}{3}, t = 0.1$	$G_i + t = 0.499026$	$G_i + t = 0.5$	$G_i + t = 0.501148$
$p_M = \frac{1}{3}, \tau = \frac{0.1}{1.5}$	$G_i + t = 0.499154$	$G_i + t = 0.5$	$G_i + t = 0.500998$

In the base case with no taxation, the symmetric equilibrium can be found as the solution to the following system:

$$\begin{aligned} g_L &= Q(w_L - g_L) - (p_L g_L + p_M g_M + p_H g_H) \\ g_M &= Q(w_M - g_M) - (p_L g_L + p_M g_M + p_H g_H) \\ g_H &= Q(w_H - g_H) - (p_L g_L + p_M g_M + p_H g_H). \end{aligned}$$

Table 1 reports individual expected contributions as p_M ranges from 1 to $1/3$ (the uniform distribution) to 0. The results accord with Proposition 4: when g is concave ($a = 3/4$ and $b = 1/4$), this increase in riskiness of income reduces contributions from 0.5 to 0.498538; when g is piecewise linear ($a = 1/2$ and $b = 1/2$), this

increase in riskiness of income leaves contributions unchanged at 0.5; and when when g is convex ($a = 1/4$ and $b = 3/4$), this increase in riskiness of income increases contributions from 0.5 to 0.501721. The first three rows of Table 1 illustrate the interpretation of Proposition 4 in which the riskiness of each agent's income distribution changes. As described after Corollary 2, what is frequently interpreted in the literature as inequality-reducing redistribution—proportional taxation with lump-sum redistribution—is also handled by Proposition 4. The fourth line of Table 1 illustrates this last interpretation. The proportional tax rate is set at $\tau = \frac{0.1}{1.5}$, so that the lump-sum transfer, l , to each player equals 0.1. The government makes up any realized shortfall and retains any realized surplus. For simplicity, we assume the government can see individuals' incomes (so there is no issue with individuals honestly reporting their incomes) and the government finances whatever realized shortfall from the expected contribution and retains any surpluses. These assumptions mean that i) there are no concerns with individuals honestly reporting incomes for tax purposes and ii) individuals can perfectly forecast the government's contribution to the public good.¹⁵ The conclusions agree with Proposition 4. In Barbieri and Malueg (2010), only the no-effect scenario illustrated in the middle column arises.

We next analyze the effects of taxation and government provision on voluntary provision of a public good. Continuing with the earlier example, we examine the uniform case ($p_M = 1/3$). If the tax is lump sum and equal to 0.1, then for the utility functions represented in Table 1, both players continue to contribute at all income levels. Therefore, there is one-for-one crowding out, and the overall level of the public good remains unchanged, relative to the case of no government intervention.

Suppose instead the government contributes the same exogenous level of the public good, namely, 0.2, this time financing it through proportional taxation at the rate $\tau = \frac{0.1}{1.5}$, so that the expected tax revenue is the same as for the lump sum taxation. Interestingly, now we see the intervention does have an effect. When g is concave, the crowding out is less than one-for-one, with an overall increase in the expected level of the public good. When g is linear, the crowding out is one-for-one, leaving the expected level of the public good unchanged. And when g is convex, the crowding out is more than one-for-one, with an overall decrease in the expected level of the public good. These results obtain because one can decompose the government intervention described above into two components: first, a public provision of the public good financed through a lump-sum tax set equal to the tax rate times expected income; and second, a reduction in the riskiness of the income distribution because taxes are not actually lump-sum, but proportional. The first component has no real effect on total provision, for the parameter values in the example, by Proposition 5.

¹⁵Of course, inducing truthful revelation of incomes for taxation purposes has long been of interest. A classic reference on this topic is Mirrlees (1971). Also, if the government's contribution were to depend on realized tax receipts, then players knowing their own income realizations but no others might hold different forecasts regarding the government's contribution. As noted above, for simplicity here we avoid such complications.

But the second component has the real effects on total provision described in Corollary 2. This explains why the results in the fourth row and sixth row of Table 1 match. In Barbieri and Malueg (2010), only the no-effect scenario illustrated in the middle column is considered.

5.2 Multidimensional private information

Our methodology can easily accommodate multidimensional private information. Beyond income, another interesting parameter of the basic pure public good model is the “unit cost” of contributions, which, in a full-information context, is analyzed in Cornes and Hartley (2007a). The possibility that agents may have different unit costs of contributions is captured by changing the budget constraint to

$$x_i + c_i g_i = w_i,$$

where c_i is the unit cost.

We now assume c_i is private information, and, for simplicity only, we assume it is uniformly distributed on $[1 - d, 1 + d]$, where d is a non-negative number smaller than 1. Running through the same calculations in Sections 3 and 4, we can rewrite the FOC in (9) as

$$Q_i(w_i - c_i g_i(w_i, c_i)) - g_i(w_i, c_i) = G_{-i},$$

and this equation defines $g_i(w_i, c_i)$ as a function of G_{-i} when $g_i(w_i, c_i)$ is positive, i.e., for $w_i > \hat{w}_i(G_{-i})$. (Note that, as in (11), the definition of the threshold value $\hat{w}_i(G_{-i})$ is not affected by the realized level of c_i .) We can denote this function by $b_i(w_i, c_i, G_{-i})$, and we have that expected contributions satisfy the following:

$$G_i = \int_{1-d}^{1+d} \int_{\hat{w}_i(G_{-i})}^{\bar{w}} \frac{1}{2d} b_i(w_i, c_i, G_{-i}) dF_i(w_i) dc_i.$$

Cornes and Hartley (2007a) analyze changes in the level of the unit costs. We can address the issue of changes in the dispersion of unit costs simply by changing d . Indeed, if we take $d_1 > d_2$, we see that the distribution of costs characterized by d_2 second-order stochastically dominates the distribution of costs characterized by d_1 . Therefore, under convexity, i.e., if

$$\int_{\hat{w}_i(G_{-i})}^{\bar{w}} \frac{\partial^2 b_i(w_i, c_i, G_{-i})}{(\partial c_i)^2} dF_i(w_i) > 0,$$

then equilibrium expected contributions under d_1 are larger than those under d_2 (and vice versa if the above displayed term is negative). This result is illustrated in the following example, which continues with

the example of this section, now taking parameters $a = 1/2$ and $b = 0$, which yields $Q_i(x) = [K_i(x) - H'_i(x)]/K'_i(x) = 2x$. And if we assume $F_1(w) = F_2(w) = w$ on $[0, 1]$, then equilibrium is characterized by

$$G_1 = \int_{1-d}^{1+d} \int_{G_2}^1 \frac{1}{2d} \frac{w_1 - G_2}{1 + c_1} dF_1(w_1) dc_1,$$

and by a similar equation for G_2 .

Noting that the integrand is convex in c_1 , we expect an increase in d to lead to larger equilibrium contributions. Indeed, this is confirmed by numerical calculations that $G_1 = G_2 \approx 0.1716$ when $d \rightarrow 0$, $G_1 = G_2 \approx 0.1722$ when $d = 0.25$, and $G_1 = G_2 \approx 0.1742$ when $d = 0.50$.

6 Conclusion

We have studied the connections between two frameworks used to analyze the private provision of a public good in the presence of private information: the binary public-good model with threshold uncertainty and the standard continuous case à la BBV. What makes the framework of the binary public good with threshold uncertainty tractable is the linearity of first-order conditions in others' contributions. We have shown that such linearity also considerably simplifies the analysis in the standard continuous case.

After identifying all utility functions that yield this linearity, we provided conditions ensuring the utility functions have the same properties that BBV require: the private good and the public good are both strictly desirable and strictly normal, and preferences are convex. Given these properties of derived demands, we established existence and uniqueness of equilibrium. Finally, we explored the issue of convexity of equilibrium strategies in income, which is private information. The curvature properties of equilibrium strategies are central to our analysis of crowding out and changes in a player's income distribution. Our analysis of how changes in a player's *ex ante* distribution of income affect individual and overall expected contributions to the public good complement BBV's analysis of redistribution of income among players.

The implications we draw when income is stochastic and strategies are *convex* align with those that BBV derive for full-information economies, but appear stronger. First, a proportional taxation with lump-sum redistribution (of the average proceeds) reduces public good provision, even if income is taxed and redistributed across contributors only. Second, crowding out of public expenditures financed through a proportional tax on contributors' income will be more than one-for-one. Both results obtain because a proportional tax reduces the riskiness of the income distribution of any agent. If the income of the first agent, who adopts a convex contribution strategy, becomes less risky, then all other agents perceive this agent's contribution as lower and they partially compensate. In turn, the first agent actually reduces his

contributions for each of his income realizations. And if income becomes symmetrically less risky for all, then all agents end up contributing less.

In contrast, when agents' contribution strategies are *concave*, our implications go in the opposite direction; indeed, we show by example that an inequality-reducing income redistribution may increase the voluntary supply of the public good, if the riskiness of individual income distributions is reduced.

Our results are useful for analyzing the effects of income inequality and public provision on the voluntary provision of public goods. If income is at least partially stochastic and its taxation is not lump sum, then taxation affects the riskiness of income distributions, with the effects we have described. If one assumes, as in BBV, that what motivates agents to give is "enlightened self-interest," then our model establishes a baseline set of implications valid when income is stochastic that cannot simply be gleaned from the full-information framework of BBV.

Our analysis provides a first look into the standard continuous case, through the tools developed for the analysis of binary public-good environments with threshold uncertainty. However, much work remains to be done to extend our results. In particular, dispensing with linearity of best responses in others' contributions is the subject of current work.¹⁶

Appendix

Proof of Proposition 1. Equation (4) is a linear partial differential equation of first order; Hildebrand (1976, p. 389) provides the general solution to this canonical form. Adapting his notation to ours, this solution is specified as follows:

$$w_2 = v_i(w_1),$$

where v_i is an arbitrary function, $w_1(x_i, \tilde{G}, u_i) = c_1$ and $w_2(x_i, \tilde{G}, u_i) = c_2$ are solutions of any two independent ordinary differential equations which imply the relationship

$$\frac{dx}{1} = \frac{d\tilde{G}}{-1} = \frac{du_i}{h_i(x_i) + k_i(x_i) \times \tilde{G}},$$

and c_1 and c_2 are arbitrary constants.

We now follow the procedure in Hildebrand's Example 1 on p. 393. Integration of $dx_i = -d\tilde{G}$, yields

$$w_1(x_i, \tilde{G}, u_i) \equiv x_i + \tilde{G} = c_1.$$

¹⁶A simulation in the Appendix provides a step in this direction, analyzing a Cobb-Douglas framework. Our numerical results on the effects of changing the riskiness of income are in accord with our analytical comparative statics.

Integration of $dx_i = du_i/(h_i(x_i) + k_i(x_i) \times \tilde{G})$, using $\tilde{G} = c_1 - x_i$, yields

$$\int_0^{x_i} [h_i(s) - k_i(s)s + k_i(s)c_1] ds = u_i + c_2;$$

substituting back for c_1 we have

$$w_2(x_i, \tilde{G}, u_i) \equiv \int_0^{x_i} [h_i(s) - k_i(s)s + k_i(s)(x_i + \tilde{G})] ds - u_i = c_2.$$

Using $w_2 = v_i(w_1)$ we obtain

$$\int_0^{x_i} [h_i(s) - k_i(s)s + k_i(s)(x_i + \tilde{G})] ds - u_i = v_i(x_i + \tilde{G});$$

hence,

$$u_i(x_i, \tilde{G}) = \int_0^{x_i} [h_i(s) + k_i(s)(x_i + \tilde{G} - s)] ds + v_i(x_i + \tilde{G}),$$

which is (5). □

Proof of Proposition 2. To demonstrate property i , note that, given (6) and $v_i \equiv 0$, we have

$$\frac{\partial u_i}{\partial x_i} = H'_i(x_i) + K'_i(x_i) \times \tilde{G}$$

and

$$\frac{\partial u_i}{\partial \tilde{G}} = K_i(x_i).$$

Therefore, because of condition 1, we have $\frac{\partial u_i}{\partial \tilde{G}} > 0$ if $x_i > 0$. Moreover, if $x_i > 0$ and $\tilde{G} > 0$, then condition 2 yields $\frac{\partial u_i}{\partial x_i} > 0$, so we have proved property i .

We now turn to the sufficiency of FOC (7). Note that

$$\begin{aligned} \frac{d^2 \mathbb{E} \left[u_i(w_i - g_i(w_i), g_i(w_i) + \sum_{j \neq i} g_j(w_j)) \mid w_i \right]}{(dg_i(w_i))^2} &= H''_i(w_i - g_i(w_i)) - 2K'_i(w_i - g_i(w_i)) \\ &\quad + K''_i(w_i - g_i(w_i)) \times (G_{-i} + g_i(w_i)) < 0, \end{aligned}$$

by conditions 2 and 3. (Below, we omit functional arguments whenever no confusion arises.)

We now verify strict normality of the private good and of the individual contribution to the public good.

In symbols, the desired normality obtains when $0 < \frac{dg_i}{dw_i} < 1$. Implicitly differentiating (7) and solving for

g'_i , we obtain:

$$g'_i = \frac{K'_i - (H''_i + K''_i \times (g_i + G_{-i}))}{2K'_i - (H''_i + K''_i \times (g_i + G_{-i}))}, \quad (15)$$

and both numerator and denominator are positive by conditions 2 and 3, so $0 < g'_i < 1$ as desired.

Finally, for strict quasiconcavity of u_i it is sufficient that

$$D \equiv \begin{vmatrix} 0 & \frac{\partial u_i}{\partial x_i} & \frac{\partial u_i}{\partial G} \\ \frac{\partial u_i}{\partial x_i} & \frac{\partial^2 u_i}{(\partial x_i)^2} & \frac{\partial^2 u_i}{(\partial x_i)(\partial G)} \\ \frac{\partial u_i}{\partial G} & \frac{\partial^2 u_i}{(\partial x_i)(\partial G)} & \frac{\partial^2 u_i}{(\partial G)^2} \end{vmatrix} > 0. \quad (16)$$

We have

$$D = \begin{vmatrix} 0 & H'_i + K'_i \tilde{G} & K_i \\ H'_i + K'_i \tilde{G} & H''_i + K''_i \tilde{G} & K'_i \\ K_i & K'_i & 0 \end{vmatrix} = 2K_i K'_i (H'_i + K'_i \tilde{G}) - K_i^2 (H''_i + K''_i \tilde{G}), \quad (17)$$

which, for any $x_i > 0$ and $\tilde{G} > 0$, is strictly positive by conditions 1, 2, and 3. \square

Proof of Proposition 3. The proof follows the methodology of Cornes and Hartley (2007a, 2007b). We begin by rewriting (12) as

$$Z_i(G_i, G) \equiv G_i - \int_{\hat{w}_i(G-G_i)}^{\bar{w}_i} b_i(w_i, G - G_i) dF_i(w_i) = 0. \quad (18)$$

Identity (18) implicitly defines G_i as a function of G . We now show that there exists a unique G consistent with equilibrium, by demonstrating that

$$\frac{\partial G_i}{\partial G} \leq 0 \quad \forall i, \quad (19)$$

with strict inequality if $G_i > 0$. To understand how uniqueness follows from (19), suppose by contradiction that two levels of total expected donations are consistent with equilibrium, G' and G'' , say, with $G'' > G'$. This implies that, for at least some i , $G_i(G'') > G_i(G')$. But this is impossible under (19).

We now proceed to prove (19). By the implicit function theorem, we have

$$\frac{\partial G_i}{\partial G} = - \frac{\frac{\partial Z_i}{\partial G}}{\frac{\partial Z_i}{\partial G_i}}.$$

Using (18), we obtain

$$\frac{\partial Z_i}{\partial G} = - \int_{\hat{w}_i(G-G_i)}^{\bar{w}_i} \frac{\partial b_i(w_i, G - G_i)}{\partial G_{-i}} dF_i(w_i),$$

since $b(\hat{w}_i(G - G_i), G - G_i) = 0$. Rewriting (9), which defines b_i where it is strictly positive, as

$$Q_i(w_i - b_i(w_i, G_{-i})) - b_i(w_i, G_{-i}) = G_{-i}, \quad (20)$$

we obtain by implicit differentiation that

$$\frac{\partial b_i}{\partial G_{-i}} [-Q'_i(w_i - b_i(w_i, G_{-i})) - 1] = 1;$$

and because $Q'_i > 0$ wherever $b_i > 0$, it now follows that

$$-1 < \frac{\partial b_i}{\partial G_{-i}} \leq 0, \quad (21)$$

with strict inequality whenever $b_i > 0$. Therefore,

$$\frac{\partial Z_i}{\partial G} \geq 0.$$

Similarly,

$$\frac{\partial Z_i}{\partial G_i} = 1 + \int_{\hat{w}_i(G - G_i)}^{\bar{w}_i} \frac{\partial b_i(w_i, G - G_i)}{\partial G_{-i}} dF_i(w_i),$$

and, using (21), we have

$$\frac{\partial Z_i}{\partial G_i} > 0.$$

Thus,

$$\frac{\partial G_i}{\partial G} = -\frac{\frac{\partial Z_i}{\partial G}}{\frac{\partial Z_i}{\partial G_i}} \leq 0,$$

with strict inequality for $G_i > 0$, which is (19).

Note now that uniqueness of G , through (18) and the above determined $\frac{\partial Z_i}{\partial G_i} \geq 0$, implies uniqueness of G_i . In turn, this implies uniqueness of G_{-i} and eventually, through (9), we obtain uniqueness for every contribution function $g_i(w_i)$.

All that is left to show is existence of an equilibrium. By the previous discussion, what is required is a value for G such that $\sum_i G_i = G$. Note that each G_i is defined as a continuous function of G through (18). Thus, $\sum_i G_i$ is also a continuous function of G . Now, using (11), define G_i^H through

$$Q_i(\bar{w}_i) = G_i^H > 0,$$

and note that $Z_i(0, G_i^H) = 0$, since $\hat{w}_i(G_i^H) = \bar{w}_i$. Thus, G_i^H is the contribution by players other than i

that just leaves player i unwilling to contribute even at his highest possible income level; so, $G_i(G_i^H) = 0$. Therefore, letting $G^H = \max_i G_i^H$, we have $\sum_i G_i(G^H) = 0 < G^H$. Consider now a player for whom assumption A3 is satisfied. Without loss of generality, take that to be $i = 1$, consider w_1 sufficiently close to \bar{w}_1 , and use (9) to define $g_1(w_1)$ through

$$g_1(w_1) = Q_1(w_1 - g_1(w_1)). \quad (22)$$

By assumptions A1–A3, (22) is satisfied for a unique $g_1(w_1) > 0$ if w_1 is close to \bar{w}_1 . To see this, note that A3 implies that the demand of agent 1 for the public good when no one else contributes is strictly positive for w_1 is close to \bar{w}_1 . Denote this level as \hat{g}_1 . By properties ii and iv in Assumption A1, \hat{g}_1 uniquely solves the FOC (7), which can be rearranged as

$$\hat{g}_1 = \frac{K_1(w_1 - \hat{g}_1) - H_1'(w_1 - \hat{g}_1)}{K_1'(w_1 - \hat{g}_1)}.$$

But since

$$Q_1(y) = \frac{K_1(y) - H_1'(y)}{K_1'(y)},$$

then $g_1(w_1) = \hat{g}_1$ is the unique solution to (22), again for w_1 sufficiently close to \bar{w}_1 .

For those values of w_1 such that (22) does not admit a positive solution, take $g_1(w_1) = 0$. Here $g_1(w_1)$ is player 1's contribution when all others contribute 0. Now, let $G^L = \int_{\bar{w}_1}^{\bar{w}_1} g_1(w_1) dF_1(w_1)$. By construction, using (18) we have $Z_1(G^L, G^L) = 0$. Therefore, $G_1(G^L) = G^L$ and *a fortiori* $\sum_i G_i(G^L) \geq G^L$. By continuity therefore, there exists some $G \in [G^L, G^H]$ such that $\sum_i G_i(G) = G$, as we wanted to show. \square

Proof of Lemma 1. Suppose Δ is such that $G_i^*(\Delta) > 0$. Let $G_{-i}^*(\Delta) \equiv \sum_{j \neq i} G_j^*(\Delta)$ denote the equilibrium contributions of all other players. With this exogenous contribution, the defining equation for player i 's equilibrium strategy, $g_i^*(w_i, \Delta)$, becomes

$$g_i^*(w_i, \Delta) = Q_i(w_i - g_i^*(w_i, \Delta)) - G_{-i}^*(\Delta) - \Delta \quad (23)$$

(this is just the obvious extension of (9)). The proof proceeds in several steps.

Step 1: $\partial g_i^*(w_i, \Delta) / \partial \Delta$ has a common sign for all w_i where $g_i^* > 0$. Differentiating (23) with respect to Δ we find, for any income level w_i where $g_i^*(w_i, \Delta) > 0$, that

$$\frac{\partial g_i^*(w_i, \Delta)}{\partial \Delta} \left[1 + Q_i'(w_i - g_i^*(w_i, \Delta)) \right] = -1 - \frac{dG_{-i}^*(\Delta)}{d\Delta}. \quad (24)$$

In (24) the term in square brackets exceeds 1 (because $Q'_i > 0$), so $\partial g_i^*/\partial \Delta$ has the same sign at all incomes where g_i^* is positive (and elsewhere it is 0).

Step 2: $dG^*(\Delta)/d\Delta \in (-1, 0)$. Rearrange (24) and take expectations with respect to w_i to obtain

$$\begin{aligned} -1 &= \frac{dG_i^*(\Delta)}{d\Delta} + \mathbb{E} \left[\frac{\partial g_i^*(w_i, \Delta)}{\partial \Delta} Q'_i(w_i - g_i^*(w_i, \Delta)) \right] + \frac{dG_{-i}^*(\Delta)}{d\Delta} \\ &= \frac{dG^*(\Delta)}{d\Delta} + T_i(\Delta), \end{aligned} \tag{25}$$

where

$$T_i(\Delta) \equiv \mathbb{E} \left[\frac{\partial g_i^*(w_i, \Delta)}{\partial \Delta} Q'_i(w_i - g_i^*(w_i, \Delta)) \right].$$

First, suppose, contrary to the claim of the step, that $dG^*(\Delta)/d\Delta \geq 0$. Then by (25) it must be that $T_i \leq -1$ for all i for all players making positive contributions. Because $Q'_i > 0$, it then follows from Step 1 that $\partial g_i^*/\partial \Delta \leq 0$ for all w_i , with strict inequality whenever $g_i^* > 0$, in turn implying $dG_i^*/d\Delta < 0$, for all players making positive contributions. But this in turn implies $dG^*/d\Delta < 0$, contradicting the assumption that $dG^*(\Delta)/d\Delta \geq 0$.

Second, suppose, contrary to the claim of the step, that $dG^*(\Delta)/d\Delta \leq -1$. Then by (25) it must be that $T_i \geq 0$ for all i for all players making positive contributions. Because $Q'_i > 0$, it then follows from Step 1 that $\partial g_i^*/\partial \Delta \geq 0$ for all w_i , with strict inequality whenever $g_i^* > 0$, in turn implying $dG_i^*/d\Delta \geq 0$ for all players making positive contributions. But this in turn implies $dG^*/d\Delta \geq 0$, contradicting the assumption that $dG^*(\Delta)/d\Delta \leq -1$.

Step 3: $dG_i^*(\Delta)/d\Delta < 0$. By (25) and Step 2 it follows that $T_i(\Delta) < 0$. Therefore, because $Q'_i > 0$, it follows from Step 1 and the definition of T_i that $\partial g_i^*(w_i, \Delta)/\partial \Delta \leq 0$, with strict inequality wherever g_i^* is positive. This in turn implies $dG_i^*/d\Delta < 0$.

Step 4: $-1 < dG_i^*(\Delta)/d\Delta < 0$. This follows from Steps 2 and 3.

Step 5: $\partial g_i^*(w_i, \Delta)/\partial \Delta \leq 0$, with strict inequality for all w_i where $g_i^* > 0$. This follows from Steps 1 and 4.

Step 6: $\frac{d\hat{w}_i(G^*(\Delta) - G_i^*(\Delta) + \Delta)}{d\Delta} > 0$. Follows by (11), $Q'_i > 0$, and the fact that $\frac{d}{d\Delta} [G^*(\Delta) - G_i^*(\Delta) + \Delta] > 0$, as established in Steps 3 and 4. \square

Proof of Proposition 5. Part 1 is a direct corollary of Lemma 1. Because only the noncontributors are taxed, the resulting contribution by the government is essentially viewed by the contributors as the exogenous contribution Δ in the lemma.

To prove part 2, we again start from equation (18), recalling that whenever $G_i(G)$ is strictly positive it satisfies

$$Z_i(G_i(G), G) \equiv G_i - \int_{\hat{w}_i(G-G_i)}^{\bar{w}_i} b_i(w_i, G - G_i) dF_i(w_i) = 0,$$

where the function $b_i(w_i, G_O)$ comes from equation (9) as the solution for g , when g is positive, of

$$Q_i(w_i - g) - g = G_O.$$

After applying the lump-sum tax t_i , the new solution, g^t , satisfies

$$Q_i(w_i - t_i - g^t) - g^t = G_O, \quad (26)$$

implying

$$g^t = b_i(w_i - t_i, G_O). \quad (27)$$

As with the derivation above of the expected contribution functions $G_i(G)$, we denote the new, after-tax individual expected contribution as $G_i^t(G)$. Therefore, from (27) we have that G_i^t , viewed as a function of total expected contributions, is

$$G_i^t = \int_{\hat{w}_i(G-G_i^t)+t_i}^{\bar{w}_i} b_i(w_i - t_i, G - G_i^t) dF_i(w_i). \quad (28)$$

We now establish two useful relations. Subtracting t_i from both sides of (26), we have, for $t_i \leq G_O$,

$$Q_i(w_i - (t_i + g^t)) - (g^t + t_i) = G_O - t_i, \quad (29)$$

implying

$$t_i + g^t = b_i(w_i, G_O - t_i). \quad (30)$$

From (27) and (30) we now obtain a first useful condition:

$$b_i(w_i - t_i, G_O) + t_i = b_i(w_i, G_O - t_i). \quad (31)$$

Second, observe as well that, if $G_i^t + t_i \leq G$, then

$$\hat{w}_i(G - (G_i^t + t_i)) \leq \hat{w}_i(G - G_i^t) < \hat{w}_i(G - G_i^t) + t_i, \quad (32)$$

where the first inequality follows from the definition of \hat{w}_i in (11) and $\hat{w}'_i(\cdot) > 0$, which derives from $Q'_i > 0$, in turn implied by the strict normality in Assumption A1 recalling the conclusions of Proposition 2.

We now investigate the value $Z_i(G_i^t + t_i, G)$. Plugging $G_i^t + t_i$ into the first argument of the definition of Z_i we have

$$\begin{aligned}
Z_i(G_i^t + t_i, G) &= (G_i^t + t_i) - \int_{\hat{w}_i(G - (G_i^t + t_i))}^{\bar{w}_i} b_i(w_i, G - (G_i^t + t_i)) dF_i(w_i) && \text{(by (18))} \\
&= t_i - \int_{\hat{w}_i(G - (G_i^t + t_i))}^{\bar{w}_i} b_i(w_i, G - (G_i^t + t_i)) dF_i(w_i) \\
&\quad + \int_{\hat{w}_i(G - G_i^t) + t_i}^{\bar{w}_i} b_i(w_i - t_i, G - G_i^t) dF_i(w_i) && \text{(by (28))} \\
&= t_i F_i(\hat{w}_i(G - G_i^t) + t_i) \\
&\quad - \int_{\hat{w}_i(G - (G_i^t + t_i))}^{\hat{w}_i(G - G_i^t) + t_i} b_i(w_i, G - (G_i^t + t_i)) dF_i(w_i) && \text{(by (31) and (32))} \\
&= t_i F_i(\hat{w}_i(G - (G_i^t + t_i))) \\
&\quad + \int_{\hat{w}_i(G - (G_i^t + t_i))}^{\hat{w}_i(G - G_i^t) + t_i} [t_i - b_i(w_i, G - (G_i^t + t_i))] dF_i(w_i). && \text{(33)}
\end{aligned}$$

Observe now that

$$\begin{aligned}
b_i(\hat{w}_i(G - G_i^t) + t_i, G - (G_i^t + t_i)) &= b_i(\hat{w}_i(G - G_i^t), G - G_i^t) + t_i && \text{(by (31))} \\
&= t_i; && \text{(by the definition of } \hat{w}_i(G - G_i^t))
\end{aligned}$$

therefore, the integrand in square brackets in (33) is strictly positive, because b_i is strictly increasing in w_i on $(\hat{w}_i(G - (G_i^t + t_i)), \hat{w}_i(G - G_i^t) + t_i)$.

There are now two cases to consider. First, if with the imposition of the tax all types of player i continue to contribute, i.e., $\hat{w}_i(G - G_i^t) + t_i \leq \underline{w}_i$, then the expression in (33) equals 0 by (32), implying $0 = Z_i(G_i^t + t_i, G)$ which, by the interpretation of equation (18) given at the beginning of this proof, yields $G_i(G) = G_i^t + t_i$, or, equivalently, $G_i^t = G_i(G) - t_i$. Thus, for those values of G and G_i^t such that $\hat{w}_i(G - G_i^t) + t_i < \underline{w}_i$, G_i^t viewed as a function of G shifts down by exactly t_i after the tax is imposed.

In the second case, some types of player i do not contribute after the tax is imposed, i.e., $\hat{w}_i(G - G_i^t) + t_i > \underline{w}_i$. In this case, (33) is strictly positive, again by (32). Therefore, $Z_i(G_i^t + t_i, G) > 0$, and because $\partial Z_i(G_i, G) / \partial G_i > 0$, it now follows that $G_i^t + t_i > G_i(G)$, which yields $G_i^t > G_i(G) - t_i$. That is, the function G_i^t shifts down by less than t_i .

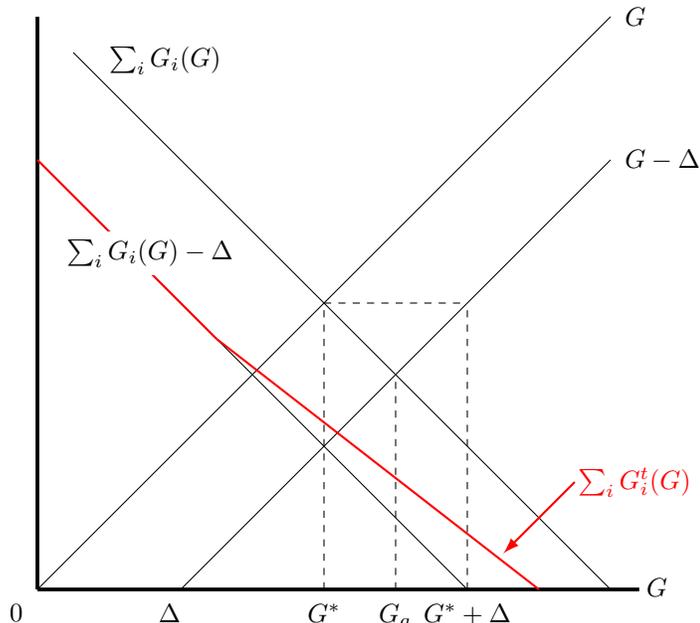


Figure 2: After taxation (t_1, \dots, t_n) , $\sum_i G_i^t(G)$ lies on or above $\sum_i G_i(G) - \Delta$.

Figure 2 is helpful in understanding the equilibrium effect of taxation followed by government contribution. Observe that taxation effects a reduction in incomes, so by normality $G_i^t(G) \leq G_i(G)$. Therefore, the foregoing analysis shows that the graph of the sum of after-tax (expected) contribution strategies $G_i^t(G)$ lies below the curve labelled $\sum_i G_i(G)$ and on or above the curve labelled $\sum_i G_i(G) - \Delta$. Moreover, the (total) equilibrium provision G^t following the introduction of taxes satisfies $G^t = \sum_i G_i^t(G^t) + \Delta$, or, equivalently, $\sum_i G_i^t(G^t) = G^t - \Delta$.

Now, if $\sum t_i = \Delta$ and after taxation all contributors remain contributors for all possible incomes, then $\sum_i G_i^t(G)$ is simply a downward shift of $\sum_i G_i(G)$ by the amount $\sum t_i = \Delta$; we have $\sum_i G_i^t(G^*) = \sum_i (G_i(G^*) - t_i) = G^* - \Delta$. Hence, the equilibrium overall supply of the public good, which solves $G - \Delta = \sum_i G_i^t(G)$, is unchanged. However, if with taxation some player at some income levels does not contribute, then $\sum_i G_i^t(G^*)$ shifts down by less than Δ , implying the post-tax equilibrium G^t will fall in the interval $(G^*, G_a]$, showing the equilibrium crowding out is only partial. The case of partial crowding out is depicted in Figure 2. \square

A Cobb-Douglas example. Here, we provide a robustness test of our results about redistribution by considering a prominent example of utility function for which our linearity condition in (4) does not apply. Nonetheless, we will show that our results on changing inequality in the distribution are confirmed by our numerical simulations.

Consider a two-player BBV game with private information. The utility function of agent i , $i = 1, 2$ is

$u_i(x_i, \tilde{G}) = x_i^\alpha \tilde{G}^\beta$, with budget constraint $w_i = x_i + \tilde{g}_i$, and $\tilde{G} = \tilde{g}_1 + \tilde{g}_2$. While we consider $\alpha > 0$ and $\beta > 0$, we here do not require that $\beta = 1$ as done previously. There is private information about income levels, with w_1 distributed according to cdf F_1 and w_2 distributed according to F_2 . Equilibrium consists of a pair of contribution functions $g_1(w_1)$ and $g_2(w_2)$.

Substituting the budget constraint into the utility function, type w_1 of agent 1 maximizes for the real number \tilde{g}_1 the following utility:

$$\int (w_1 - \tilde{g}_1)^\alpha (\tilde{g}_1 + g_2(w_2))^\beta dF_2(w_2); \quad (34)$$

the FOC is

$$\int r_1(\tilde{g}_1, g_2(w_2), w_1) dF_2 \leq 0, \quad (35)$$

with equality if $\tilde{g}_1 > 0$, and where $r_1(\tilde{g}_1, g_2(w_2), w_1) \equiv \left(\frac{\beta}{\alpha} w_1 - \frac{\alpha+\beta}{\alpha} \tilde{g}_1 - g_2(w_2)\right) (\tilde{g}_1 + g_2(w_2))^{\beta-1}$.

In our simulations below, we consider a common, symmetric distribution of income for agents 1 and 2. In particular, we suppose there are only three possible income levels, $w_L = 1$, $w_M = 1.5$, and $w_H = 2$. The associated probabilities are symmetric about the expected income of 1.5: $\Pr(w_i = w_M) = p_M$ and $\Pr(w_i = w_L) = \Pr(w_i = w_H) = (1 - p_M)/2$. Thus, reductions in p_M yield riskier distributions of income. For simplicity, the examples are such that players contribute at all possible income levels. Therefore, the numbers below are derived using (35) with equality.¹⁷

Table 2 reports individual expected contributions as p_M ranges from 0.9 to 0.5 to 0.1. The results accord with Proposition 4: when g is concave ($\alpha = 3/4$ and $\beta = 1/2$), this increase in riskiness of income increases contributions from 0.3760 to 0.3842; and when g is convex ($\alpha = 5/4$ and $\beta = 2$), this increase in riskiness of income reduces contributions from 0.6647 to 0.6465. \square

References

- Admati, Anat R., and Motty Perry (1991), Joint projects without commitment, *Review of Economic Studies* 58, 259–276.
- Alboth, Dirk, Anat Lerner, and Jonathan Shalev (2001), Profit maximizing in auctions of public goods, *Journal of Public Economic Theory* 3, 501–525.
- Andreoni, James (2006), Leadership giving in charitable fund-raising, *Journal of Public Economic Theory* 8, 1–22.

¹⁷We have also verified that the expected utility in (34) is strictly quasiconcave, for the relevant parameter values and derived solutions. Details are available upon request.

Table 2: Voluntary contributions

	$\alpha = \frac{5}{4}, \beta = 2$ ($g(\cdot)$ is convex)
$p_M = 0.9$	$g(w_L) = 0.3562, g(w_M) = 0.6647, g(w_H) = 0.9729$ $G_1 = G_2 = 0.6647$
$p_M = 0.5$	$g(w_L) = 0.3444, g(w_M) = 0.6565, g(w_H) = 0.9669$ $G_1 = G_2 = 0.6561$
$p_M = 0.1$	$g(w_L) = 0.3317, g(w_M) = 0.6480, g(w_H) = 0.9608$ $G_1 = G_2 = 0.6465$
	$\alpha = \frac{3}{4}, \beta = \frac{1}{2}$ ($g(\cdot)$ is concave)
$p_M = 0.9$	$g(w_L) = 0.1767, g(w_M) = 0.3760, g(w_H) = 0.5757$ $G_1 = G_2 = 0.3760$
$p_M = 0.5$	$g(w_L) = 0.1829, g(w_M) = 0.3798, g(w_H) = 0.5781$ $G_1 = G_2 = 0.3802$
$p_M = 0.1$	$g(w_L) = 0.1883, g(w_M) = 0.3833, g(w_H) = 0.5804$ $G_1 = G_2 = 0.3842$

Andreoni, James, and Ravi Kanbur (2007), Guest editor's introduction: Twenty surprising years of BBV, *Journal of Public Economics* 91, 1643–1644.

Angeletos, George-Marios, and Alessandro Pavan (2007), Efficient use of information and social value of information, *Econometrica* 75, 1103–1142.

Bag, Parimal Kant, and Santanu Roy (2008), Repeated charitable contributions under incomplete information, *The Economic Journal* 118, 60–91.

Bag, Parimal Kanti, and Santanu Roy (2011), On sequential and simultaneous contributions under incomplete information, *International Journal of Game Theory* 40, 119–145.

Barbieri, Stefano, and David A. Malueg (2008a), Private provision of a discrete public good: continuous-strategy equilibria in the private-information subscription game, *Journal of Public Economic Theory* 10, 529–545.

Barbieri, Stefano, and David A. Malueg (2008b), Private provision of a discrete public good: efficient equilibria in the private-information contribution game, *Economic Theory* 37, 51–80.

- Barbieri, Stefano, and David A. Malueg (2010), Threshold uncertainty in the private-information subscription game, *Journal of Public Economics* 94, 848–861.
- Barbieri, Stefano, and David A. Malueg (2014), Increasing fundraising success by decreasing donors' choice, *Journal of Public Economic Theory* 16, 372–400.
- Bergstrom, Theodore, Lawrence Blume, and Hal Varian (1986), On the private provision of public goods, *Journal of Public Economics* 29, 25–49.
- Cornes, Richard, and Roger Hartley (2007a), Aggregative public good games, *Journal of Public Economic Theory* 9, 201–219.
- Cornes, Richard, and Roger Hartley (2007b), Weak links, good shots and other public good games: Building on BBV, *Journal of Public Economics* 91, 1684–1707.
- Cornes, Richard, and Todd Sandler (1985), The simple analytics of pure public good provision, *Economica* 52, 103–116.
- Domar, Evsey, and Richard Musgrave (1944), Proportional Income Taxation and Risk-Taking, *Quarterly Journal of Economics* 58, 388–422.
- Gradstein, Mark, Shmuel Nitzan, and Steven Slutsky (1993), Neutrality and the private provision of public goods with incomplete information, *Economics Letters* 46, 69–75.
- Hermalin, Benjamin (1998), An economic theory of leadership, *American Economic Review* 88, 1188–1206.
- Hildebrand, Francis B. (1976): *Advanced Calculus for Applications*, 2nd. ed., Prentice-Hall Inc., Englewood Cliffs, New Jersey.
- Kanbur, S. M. (1981), Risk taking and taxation: An alternative perspective, *Journal of Public Economics* 15, 163–184.
- Kobayashi, Hajime, and Hideo Suehiro (2005), The emergence of leadership in teams, *Japanese Economic Review* 56, 296–316.
- Komai, Mana, and Mark Stegeman (2010), Leadership based on asymmetric information, *RAND Journal of Economics* 41, 35–63.
- Komai, Mana, Mark Stegeman, and Benjamin Hermalin (2007), Leadership and information, *American Economic Review* 97, 944–947.

- Krasteva, Silvana, and Huseyin Yildirim (2013), (Un)informed charitable giving, *Journal of Public Economics* 106, 14–26.
- Laussel, Didier, and Thomas R. Palfrey (2003), Efficient equilibria in the voluntary contributions mechanism with private information, *Journal of Public Economic Theory* 5, 449–478.
- Martimort, David, and Humberto Moreira (2010), Common agency and public good provision under asymmetric information, *Theoretical Economics* 5, 159–213.
- McClelland, Robert and Arthur C. Brooks (2004), What is the real relationship between income and charitable giving? *Public Finance Review* 32, 483–497.
- Menezes, Flavio M., Paulo K. Monteiro, and Akram Temimi (2001), Private provision of discrete public goods with incomplete information, *Journal of Mathematical Economics* 35, 493–514.
- Mirrlees, James A. (1971), An exploration in the theory of optimum income taxation, *The Review of Economic Studies* 38, 175–208.
- Nitzan, Shmuel and Richard E. Romano (1990), Private provision of a discrete public good with uncertain cost, *Journal of Public Economics* 42, 357–370.
- Palfrey, Thomas R., and Howard Rosenthal (1984), Participation and the provision of discrete public goods: a strategic analysis, *Journal of Public Economics* 24, 171–193.
- Palfrey, Thomas R., and Howard Rosenthal (1988), Private incentives in social dilemmas: The effects of incomplete information and altruism, *Journal of Public Economics* 35, 309–332.
- Payne, Abigail A., and Justin Smith (2014), Does income inequality increase charitable giving? *Canadian Journal of Economics*, forthcoming.
- Persson, Torsten, and Guido Tabellini (1994), Is Inequality Harmful for Growth? *The American Economic Review* 84, 600–621.
- Varian, Hal (1994), Sequential contributions to public goods, *Journal of Public Economics* 53, 165–183.
- Vesterlund, Lise (2003), The informational value of sequential fundraising, *Journal of Public Economics* 87, 627–657.
- Warr, Peter (1983), The private provision of a public good is independent of the distribution of income, *Economic Letters* 13, 207–211.